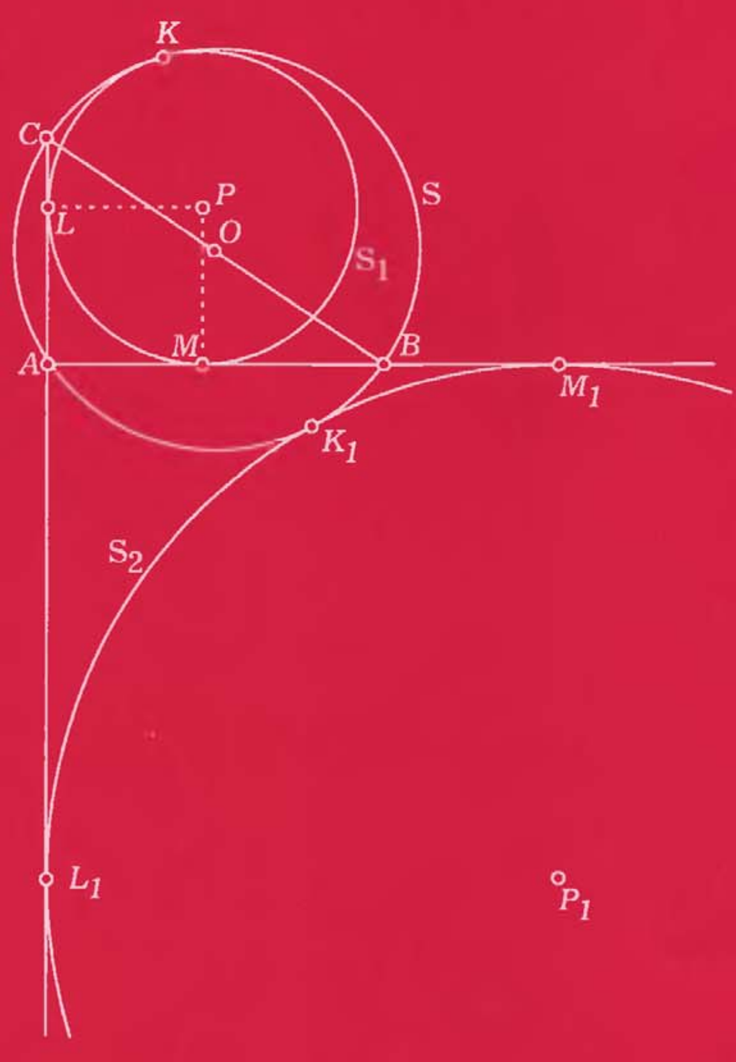
Problem Primer for the Olympiad



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Preface to the second edition

We are very happy at the enthusiastic response from students and teachers to *Problem Primer for the Olympiad*. After six years and several reprints we felt it was time to revise and include more problems. We have added more problems in each topic and have enlarged the section containing practice problems. We have corrected several misprints / errors which were brought to our notice by diligent readers to whom we are grateful. Another major change in this edition is that we have redrawn all the figures using the graphics package *GPL* developed by A. Kumaraswamy of the Rishi Valley School. We thank him for allowing us to make use of this package. We would be grateful if any errors and inaccuracies that may still remain are brought to our notice.

December 2000

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C S Yogananda

Preface

Mathematical problems are integral to any culture as much as music or other forms of art. Here is a charming example from Bhaskaracharya's *Lilavati* (Circa 1140 A.D.):

> A peacock perched on the top of a nine foot high pillar sees a snake, three times as distant from the pillar as the height of the pillar, sliding towards its hole at the bottom of the pillar. The peacock immediately flies to grab the snake. If the speeds of the peacock's flight and the snake's slide are equal, at what distance from the pillar will the peacock grab the snake?

The "three famous problems of antiquity" (namely the trisection of an angle, the doubling of the cube and the squaring of a circle, using only straight edge and compasses) from the Greek civilisation, the Temple Geometry problems of more recent origin(17th and 18th centuries) from Japan are other examples that come to mind. Problems as puzzles are found in the oldest written records.

However, use of problems as a means of searching for talent at school level is one of recent origin, probably beginning with the Eotvos competitions in Hungary which started in 1894. Though Mathematical contests / Olympiads have been a regular annual feature in many places in India (for instance, Andhra Pradesh, Bihar, Gujarat, Karnataka) for many years now, they received fresh impetus in 1986 when the National Board for Higher Mathematics (NBHM) organised the first Indian National Mathematical Olympiad (INMO). One of the main purposes of the INMO was to gauge the level of the available talent to see if India could send a team to International Mathematical Olympiad (IMO). The IMO, which began in 1959 is an annual event and is becoming more and more popular everywhere. The results of the INMO were very encouraging and India has been regular participant in the IMO since 1989. Participation in the INMO is by invitation. The top 30 or 40 students at the Regional Mathematical Olympiads (RMO) which are held in various regions are invited to participate in the INMO. The RMOs are open to students of Class 11 and below. Both INMO and RMO are written examinations consisting of 6 -10 problems. The problems are of a very different nature from the problems students usually encounter in their school curriculum. These problems are designed to challenge the students and bring the best in them to the fore. Moreover, the intensive preoccupation with interesting problems of simple and elementary nature and the effort of finding complete and elegant solutions give the students new experience, the taste of creative intellectual adventure. It is the very fond hope of all associated with the Olympiad activity that this will induce them to take up Mathematics as a career.

A major obstacle to students preparing to take part in the Olympiad is the scarcity of 'Problem literature'. This motivated us, as a first step, to compile problems that have appeared in the previous RMOs and INMOs along with detailed solutions. Our future projects in this direction include 'Problem Newsletter' and more problem compilations. A list of all the regional co-ordinators is given at the end of the book. You could approach them for information regarding the RMOs in your region.

Though we have made every effort to make the book free of errors-printing or mathematical - we might not have succeeded completely. We would be grateful to the readers who would bring to our notice any inaccuracies which may still remain.

To the student readers

The best way to use this book is, of course, to look up the problems and solve them! If you cannot get started then look up the section 'Tool Kit' which is a collection of theorems and results which are generally not available in school textbooks but which are extremely useful in solving problems. As in any other trade, you will have to familiarize yourself with the tools and understand them to be able to use them effectively. We strongly recommend that that you try to devise your own proofs for these results or look these up in books. (A list of reference books is given at the end.) You should look up the solution only after you have tried the problem on your own for some time. And the story does not end with the complete solution of a problem - you look for other solutions, generalisations, interconnections between various problems, ... One of our experiences is a perfect case in point. The problems 51 and 65 in this book appeared in the INMO 92. Our first solutions to both these problems used simple, straight-forward trigonometry. Later, it turned out that there were elegant 'pure geometry' solutions to both, and in fact the two problems were related! We leave it as a challenge to you to use Problem 51 to get another 'pure geometry' solution to problem 65. With this challenge we leave you to start on your voyage of discovery! May you come up with many gems and when you do, be sure to let us know.

Acknowledgments

We wish to sincerely thank

S. A. Shirali, who got us out of the difficulty whenever we got stuck and who took keen interest in this book-from the 'idea' to the final product;

Kahlon, Krishnan and R. Subramanian whose elegant solutions to some of the problems have been reproduced here;

Izhar Hussain and Phoolan Prasad for inspiration and guidance;

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the national board of higher mathematics (DAE).

"And a final observation. We should not forget that the solution of any worth-while problem very rarely comes to us easily and without hard work; it is rather the result of intellectual effort of days or weeks or months. Why should the young mind be willing to make this supreme effort? The explanation is probably the instinctive preference for certain values, that is, the attitude which rates intellectual effort and spiritual achievement higher than material advantage. Such a valuation can only be the result of long cultural development of environment and public spirit which is difficult to accelerate by the governmental aid or even by more intensive training in mathematics. The most effective means may consist of transmitting to the young mind the beauty of intellectual work and the feeling of satisfaction following a great and successful mental effort ."

Gabor Szego

List of symbols

$a \mid b$	a divides b.
(a, b)	greatest common divisor
$[x]$	integer part of x, i.e., the greatest integer less than or equal to x.
$n!$	(read as '*n factorial*') $= 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.
$\binom{n}{r}$ or ${}^nC_r$	the binomial coefficient; the number of combinations of n things taken r at a time $= \dfrac{n!}{r!(n-r)!}$.
$\sum_{i=1}^{n} a_i$	the sum $a_1 + a_2 + \cdots + a_n$.
$\prod_{i=1}^{n} a_i$	the product $a_1 a_2 \cdots a_n$.
$\lvert A \rvert$	cardinality of a set A, i.e., the number of elements in A.
$[ABC]$	area of triangle ABC.
$f \circ g$	composite of the functions f and g; $f \circ g(x) = f(g(x))$.

Chapter 1

Problems

1.1 Number Theory

1. Find the least number whose last digit is 7 and which becomes 5 times larger when this last digit is carried to the beginning of the number.

2. All the 2-digit numbers from 19 to 93 are written consecutively to form the number $N = 19202122\ldots919293$. Find the largest power of 3 that divides N.

3. If x, y, z and $n > 1$ are natural numbers with $x^n + y^n = z^n$ then show that x, y and z are all greater than n.

4. Given two relatively prime integers m and n, both greater than 1, show that
$$\frac{\log_{10} m}{\log_{10} n}$$
is not a rational number.

5. If a, b, x and y are integers greater than 1 such that a and b have no common factors except 1 and $x^a = y^b$, show that $x = n^b$ and $y = n^a$ for some integer n greater than 1.

6. Determine all pairs (m, n) of positive integers for which $2^m + 3^n$ is a square.

7. Prove that $n^4 + 4^n$ is not a prime number for any integer $n > 1$.

8. Find all four - digit numbers having the following properties:

    i. it is a square,

    ii. its first two digits are equal to each other and

    iii. its last two digits are equal to each other.

9. If a, b, c are any three integers then show that

$$abc(a^3 - b^3)(b^3 - c^3)(c^3 - a^3)$$

is divisible by 7.

10. Determine the largest 3 - digit prime factor of the integer $\binom{2000}{1000}$.

11. If

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$$

where a, b, c are positive integers with no common factor, prove that $(a + b)$ is a square.

12. Show that there is a natural number n such that $n!$ when written decimal notation (that is, in base 10) ends exactly in 1993 zeroes.

13. Find the remainder when 2^{1990} is divided by 1990.

14. Determine all non-negative integral pairs (x, y) for which

$$(xy - 7)^2 = x^2 + y^2.$$

15. Determine with proof, all the positive integers n for which

    i. n is not the square of any integer and

    ii. $[\sqrt{n}]^3$ divides n^2.

($[x]$ denotes the largest integer that is less than or equal to x.)

16. Prove that the product of 4 consecutive natural numbers cannot be a cube.

17. i. Determine the set of positive integers n for which 3^{n+1} divides $2^{3^n} + 1$.

    ii. Prove that 3^{n+2} does not divide $2^{3^n} + 1$ for any positive integer n.

18. For any positive integer n, let $s(n)$ denote the number of ordered pairs (x, y) of positive integers for which

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}.$$

For instance, $s(2) = 3$. Find the set of positive integers n for which $s(n) = 5$.

19. For a positive integer n, define $A(n)$ to be

$$\frac{(2n)!}{(n!)^2}.$$

Determine the sets of positive integers n for which:

i $A(n)$ is an even number;

ii $A(n)$ is a multiple of 4.

20. Show that there are infinitely many positive integers A such that $2A$ is a square, $3A$ is a cube and $5A$ is a fifth power.

21. Find all prime numbers p for which there are integers x, y satisfying

$$p + 1 = 2x^2 \text{ and } p^2 + 1 = 2y^2.$$

22. Find all triples (a, b, c) of positive integers such that

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) = 3.$$

23. Given any positive integer n show that there are two positive rational numbers a and b, $a \neq b$, which are *not* integers and which are such that $a - b, a^2 - b^2, a^3 - b^3, \ldots, a^n - b^n$ are all integers.

24. Find all primes p for which the quotient $(2^{p-1} - 1)/p$ is a square.

25. Solve for integers x, y, z:

$$x + y = 1 - z, \qquad x^3 + y^3 = 1 - z^2.$$

1.2 Algebra

26. Determine the largest number in the infinite sequence $1, \sqrt[2]{2}, \sqrt[3]{3}, \ldots, \sqrt[n]{n}, \ldots$.

27. If a, b and c are odd integers, prove that the roots of the quadratic equation $ax^2 + bx + c = 0$ cannot be rational numbers.

28. If a and b are positive real numbers such that $a + b = 1$, prove that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}.$$

29. Show that there do not exist any distinct natural numbers a, b, c and d such that

$$a^3 + b^3 = c^3 + d^3 \quad \text{and} \quad a + b = c + d.$$

30. If $a_0, a_1, \ldots\ldots a_{50}$ are the coefficients of the polynomial

$$(1 + x + x^2)^{25},$$

prove that the sum $a_0 + a_2 + \ldots + a_{50}$ is even.

31. Prove that the polynomial

$$f(x) = x^4 + 26x^3 + 52x^2 + 78x + 1989$$

cannot be expressed as product $f(x) = p(x)q(x)$ where $p(x), q(x)$ are both polynomials with integral coefficients and with degree not more than 3.

32. If a, b, c and d are any four real numbers, not all equal to zero, prove that the roots of the equation $x^6 + ax^3 + bx^2 + d = 0$ cannot all be real.

33. Given that the equation $x^4 + px^3 + qx^2 + rx + s = 0$ has four real positive roots, prove that

    (i) $pr - 16s \geq 0$,

    (ii) $q^2 - 36s \geq 0$,

    where equality holds, in each case, if and only if the four roots are equal.

34. Let a, b, c be real numbers with $0 < a < 1, 0 < b < 1, 0 < c < 1$ and $a + b + c = 2$. Prove that

$$\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8.$$

35. Prove that

$$1 < \frac{1}{1001} + \frac{1}{1002} + \frac{1}{1003} + \cdots + \frac{1}{3001} < \frac{4}{3}.$$

36. If x, y and z are three real numbers such that

$$x + y + z = 4 \quad \text{and} \quad x^2 + y^2 + z^2 = 6,$$

then show that each of x, y, z lie in the closed interval $[(2/3), 2]$. Can x attain the extreme values $2/3$ and 2?

37. Let $f(x)$ be a polynomial with integer coefficients. Suppose for five distinct integers a_1, a_2, a_3, a_4 and a_5 one has $f(a_i) = 2$ for $1 \leq i \leq 5$. Show that there is no integer b such that $f(b) = 9$.

38. Determine all functions $f : \mathbf{R} \setminus \{0, 1\} \to \mathbf{R}$ (here $\mathbf{R}$ denotes the set of real numbers) satisfying the functional relation

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}, \text{ for } x \neq 0 \text{ and } x \neq 1.$$

39. Let $p(x) = x^2 + ax + b$ be a quadratic polynomial in which a and b are integers. Given any integer n, show that there is an integer M such that $p(n)p(n+1) = p(M)$.

40. If $a_1, a_2, \ldots, a_n$ are n distinct odd natural numbers not divisible by any prime greater than 5, show that

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n} < 2.$$

41. If $p(x)$ is a polynomial with integer coefficients and a, b, c are three distinct integers, then show that it is impossible to have $p(a) = b$, $p(b) = c$ and $p(c) = a$.

42. Let a, b, c denote the sides of a triangle, show that the quantity

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

lies between the limits $3/2$ and 2. Can equality hold at either limit?

43. Let f be a function defined on the set of non-negative integers and taking values in the same set. Suppose we are given that

   i. $x - f(x) = 19\left[\frac{x}{19}\right] - 90\left[\frac{f(x)}{90}\right]$ for all non-negative integers x;

   ii. $1900 < f(1900) < 2000$.

   Find all the possible values of $f(1900)$. (Here $[z]$ denotes the largest integer $\leq z$; e.g., $[3.145] = 3$.)

44. For positive real numbers a, b, c, d satisfying $a + b + c + d \leq 1$ prove the following inequality:

$$\frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c} \leq \frac{1}{64abcd}.$$

45. Find all cubic polynomials $p(x)$ such that $(x-1)^2$ is a factor of $p(x) + 2$ and $(x+1)^2$ is a factor of $p(x) - 2$.

46. Solve for real x:

$$\frac{1}{[x]} + \frac{1}{[2x]} = \{x\} + \frac{1}{3},$$

where $[x]$ is the greatest integer less than or equal to x and $\{x\} = x - [x]$. [e.g., [3.4]=3 and {3.4}=0.4].

47. If p, q, r are the roots of the cubic equation

$$x^3 - 3px^2 + 3q^2x - r^3 = 0$$

show that $p = q = r$.

48. Define a sequence $\langle a_n \rangle_{n\geq 1}$ by

$$a_1 = 1,\ a_2 = 2 \quad \text{and} \quad a_{n+2} = 2a_{n+1} - a_n + 2,\ n \geq 1.$$

Prove that for any m, $a_m a_{m+1}$ is also a term in the sequence.

49. Suppose a and b are two positive real numbers such that the roots of the cubic equation $x^3 - ax + b = 0$ are all real. If α is a root of this cubic with minimal absolute value prove that

$$\frac{b}{a} < \alpha \leq \frac{3b}{2a}.$$

50. Let a, b, c be three real numbers such that $1 \geq a \geq b \geq c \geq 0$. Prove that if λ is a root of the cubic equation $x^3 + ax^2 + bx + c = 0$ (real or complex), then $|\lambda| \leq 1$.

1.3 Geometry

51. In a triangle ABC, $\angle A$ is twice $\angle B$. Show that $a^2 = b(b + c)$. (In fact, the converse is also true. Prove it.)

52. Two circles C_1 and C_2 intersect at two distinct points P and Q in a plane. Let a line passing through P meet the circles C_1 and C_2 in A and B respectively. Let Y be the midpoint of AB and QY meet the circles C_1 and C_2 in X and Z respectively. Show that Y is also the midpoint of XZ.

53. Suppose $ABCD$ is a cyclic quadrilateral and the diagonals AC and BD intersect at P. Let O be the circumcentre of triangle APB and H the orthocentre of triangle CPD. Show that O, P, H, are collinear.

54. Given a triangle ABC in a plane Σ find the set of all points P lying in the plane Σ such that the circumcircles of triangles ABP, BCP and CAP are congruent.

55. Suppose $ABCD$ is a cyclic quadrilateral and x, y, z are the distances of A from the lines BD, BC, BD respectively. Prove the

$$\frac{BD}{x} = \frac{BC}{y} + \frac{CD}{z}.$$

56. Suppose $ABCD$ is a convex quadrilateral and P, Q are the midpoints of CD, AB. Let AP, DQ meet in X and BP, CQ meet in Y. Prove that $[ADX] + [BCY] = [PXQY]$. How does the conclusion alter if $ABCD$ is not a convex quadrilateral?

57. Suppose P is an interior point of a triangle ABC and AP, BP, CP meet the opposite sides BC, CA, AB in D, E, F respectively. Show that

$$\frac{AF}{FB} + \frac{AE}{EC} = \frac{AP}{PD}.$$

58. Two circles with radii a and b touch each other externally. Let c be the radius of the circle that touches these two circles externally as well as a common tangent to the two circles. Prove that

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}.$$

59. Construct a triangle ABC given h_a, h_b, the altitudes from A and B and m_a, the median through A.

60. Given the angle QBP and a point L outside the angle QBP, draw a straight line through L meeting BQ in A and BP in C such that the triangle ABC has a given perimeter.

61. Triangle ABC has incentre I and the incircle touches BC, CA at D, E respectively. If BI meets DE in G, show that AG is perpendicular to BG.

62. Let A be one of the two points of intersection of two circles with centres X and Y. The tangents at A to these two circles meet the circles again at B, C. Let the point P be located so that $PXAY$ is a parallelogram. Show that P is the circumcentre of triangle ABC.

63. A triangle ABC has incentre I. Points X, Y are located on the line segments AB, AC respectively so that $BX \cdot AB = IB^2$ and $CY \cdot AC = IC^2$. Given that X, I, Y are collinear, find the possible values of the measure of angle A.

64. A triangle ABC has incentre I. Its incircle touches the side BC at T. The line through T parallel to IA meets the incircle at S and the tangent to the incircle at S meets sides AB, AC in points C', B' respectively. Prove that triangle $AB'C'$ is similar to triangle ABC.

65. Suppose $A_1A_2A_3 \ldots A_n$ is an n-sided regular polygon such that

$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$$

Determine n, the number of sides of the polygon.

66. Suppose $ABCD$ is a quadrilateral such that a semicircle with its centre at the midpoint of AB and bounding diameter lying on AB touches the other three sides BC, CD and DA. Show that

$$AB^2 = 4BC \cdot AD.$$

67. Let ABC be an acute-angled triangle. For any point P lying within this triangle, let D, E, F denote the feet of the perpendiculars from P onto the sides of BC, CA, AB respectively. Determine the set of all possible positions of the point P for which the triangle DEF is isosceles. For which positions of P will the triangle DEF become equilateral.

68. Three congruent circles have a common point O and lie inside a triangle such that each circle touches a pair of sides of the triangle. Prove that the incentre and the circumcentre of the triangle and the point O are collinear.

69. Let ABC be a triangle with $\angle A = 90°$, and S be its circumcircle. Let S_1 be the circle touching the rays AB, AC and the circle S internally. Further let S_2 be the circle touching the rays AB, AC and the circle S externally. If r_1, r_2 be the radii of the circles S_1 and S_2 respectively, show that $r_1 r_2 = 4[ABC]$.

70. The diagonals AC and BD of a cyclic quadrilateral $ABCD$ meet at right angles in E. Prove that

$$EA^2 + EB^2 + EC^2 + ED^2 = 4R^2,$$

where R is the radius of the circumscribing circle.

71. Suppose $ABCD$ is a rectangle and P, Q, R, S are points on the sides AB, BC, CD, DA respectively. Show that

$$PQ + QR + RS + SP \geq \sqrt{2}\, AC.$$

72. Let P be an interior point of an equilateral triangle ABC such that $AP^2 = BP^2 + CP^2$. Prove that $\angle BPC = 150°$.

73. Let ABC be a triangle and h_a be the altitude through A. Prove that

$$(b + c)^2 \geq a^2 + 4h_a^2.$$

(As usual a, b, c denote the sides BC, CA, AB respectively.)

74. Let P be an interior point of a triangle ABC and let BP and CP meet AC and AB in E and F respectively. If $[BPF] = 4$, $[BPC] = 8$ and $[CPE] = 13$, find $[AFPE]$. (Here [ ] denotes the area of a triangle or a quadrilateral as the case may be.)

75. Suppose $ABCD$ is a cyclic quadrilateral inscribed in a circle of radius one unit. If

$$AB \cdot BC \cdot CD \cdot DA \geq 4,$$

prove that $ABCD$ is a square.

1.4 Combinatorics

76. Consider the collection of all three-element subsets drawn from the set $\{1, 2, 3, 4, \ldots, 299, 300\}$. Determine the number of these subsets for which the sum of the three elements is a multiple of 3.

77. How many 3-element subsets of the set $\{1, 2, 3, \ldots, 19, 20\}$ are there such that the product of the three numbers in the subset is divisible by 4?

78. Suppose $A_1, A_2, \ldots, A_6$ are six sets each with four elements and $B_1, B_2, \ldots, B_n$ are n sets each two elements such that

$$A_1 \cup A_2 \cup \ldots \cup A_6 = B_1 \cup B_2 \cup \ldots \cup B_n = S \text{ (say)}.$$

Given that each element of S belongs to exactly four of the A_i's and exactly three of the B_j's, find n.

79. Two boxes contain between them 65 balls of several different sizes. Each ball is white, black, red, or yellow. If you take any five balls of the same colour, at least two of them will always be of the same size (radius). Prove that there are at least three balls which lie in the same box, have the same colour and are of the same size.

80. There are two urns each containing an arbitrary number of balls. Both are non empty to begin with. We are allowed two types of operations:

    a. Remove an equal number of balls simultaneously from both urns;

    b. Double the number of balls in any one them.

    Show that after performing these operations finitely many times, both the urns can be made empty.

81. Let A denote a subset of the set $\{1, 11, 21, 31, \ldots 541, 551\}$ having the property that no two elements of A add upto 552. Prove that A cannot have more than 28 elements.

82. Let $A = \{1, 2, 3, \ldots 100\}$ and B a subset of A having 48 elements. Show that B has two distinct elements x and y whose sum is divisible by 11.

83. Find the number of permutations, $(P_1, P_2, \ldots, P_6)$, of $(1, 2, \ldots, 6)$ such that for any k, $1 \leq k \leq 5$, $(P_1, P_2, \ldots, P_k)$ does not form a permutation of $1, 2, \ldots, k$.

    [That is, $P_1 \neq 1$; (P_1, P_2) is not a permutation of $1, 2, 3$, etc.]

84. There are seventeen distinct positive integers such that none of them has a prime factor exceeding 10. Show that the product of some two of them is a square.

85. Let A be a subset of $\{1, 2, 3, \ldots, 2n-1, 2n\}$ containing $n+1$ elements. Show that

    a. Some two elements of A are relatively prime:
    b. Some two elements of A have the property that one divides the other.

86. Given seven arbitrary distinct real numbers, show that there exist two numbers x and y such that

$$0 < \frac{x - y}{1 + xy} < \frac{1}{\sqrt{3}}.$$

87. There are six cities in an island and every two of them are connected either by train or by bus, but not by both. Show that there are three cities which are mutually connected by the same mode of transport.

88. There are eight points in the plane such that no three of them are collinear. Find the maximum number of triangles formed out of these points such that no two triangles have more than one vertex in common.

89. How many increasing 3-term geometric progressions can be obtained from the sequence $1, 2, 2^2, 2^3, \ldots, 2^n$?
    (e.g., $\{2^2, 2^5, 2^8\}$ is a 3-term geometric progression for $n \geq 8$.)

90. Let A denote the set of all numbers between 1 and 700 which are divisible by 3 and let B denote the set of all numbers between 1 and 300 which are divisible by 7. Find the number of all ordered pairs (a, b) such that $a \in A, b \in B, a \neq b$ and $a + b$ is even.

91. If $A \subset \{1, 2, 3, \ldots, 100\}$, $|A| = 50$ such that no two numbers from A have their sum as 100 show that A contains a square.

92. Find the number of unordered pairs $\{A, B\}$ (i.e., the pairs $\{A, B\}$ and $\{B, A\}$ are considered to be the same) of subsets of an n-element set X which satisfy the conditions:

   (a) $A \neq B$;
   (b) $A \cup B = X$.

93. Find the minimum possible least common multiple (l.c.m.) of twenty (not necessarily distinct) natural numbers whose sum is 801.

94. Find the number of quadratic polynomials, $ax^2 + bx + c$, which satisfy the following conditions:

   (a) a, b, c are distinct;
   (b) $a, b, c \in \{1, 2, 3, \ldots, 1999\}$ and
   (c) $x + 1$ divides $ax^2 + bx + c$.

95. Show that the number of 3-element subsets $\{a, b, c\}$ of the set $\{1, 2, 3, \ldots, 63\}$ with $a + b + c < 95$ is less than the number of those with $a + b + c > 95$.

96. Let X be a set containing n elements. Find the number of all ordered triples (A, B, C) of subsets of X such that A is a subset of B and B is a *proper* subset of C.

97. Find the number of 4×4 arrays whose entries are from the set $\{0, 1, 2, 3\}$ and which are such that the sum of the numbers in each of the four rows and in each of the four columns is divisible by 4. (An $m \times n$ array is an arrangement of mn numbers in m rows and n columns.)

98. There is a $2n \times 2n$ array (matrix) consisting of 0's and 1's and there are exactly $3n$ zeros. Show that it is possible to remove all the zeros by deleting some n rows and some n columns.

99. For which positive integral values of n can the set $\{1, 2, 3, \ldots, 4n\}$ be split into n disjoint 4-element subsets $\{a, b, c, d\}$ such that in each of these sets $a = (b + c + d)/3$.

100. For any natural number n, ($n \geq 3$), let $f(n)$ denote the number of non-congruent integer-sided triangles with perimeter n (e.g., $f(3) = 1, f(4) = 0, f(7) = 2$). Show that

(a) $f(1999) > f(1996)$; (b) $f(2000) = f(1997)$.

1.5 Miscellaneous

101. The sixty-four squares of a chess board are filled with positive integers one on each in such a way that each integer is the average of the integers on the neighbouring squares. (Two squares are neighbours if they share a common edge or vertex. Thus a square can have 8, 5 or 3 neighbours depending on its position).
Show that all the sixty-four entries are in fact equal.

102. Let T be the set of all triples (a, b, c) of integers such that $1 \leq a < b < c \leq 6$. For each triple (a, b, c) in T, take the product abc. Add all these products corresponding to all triples in T. Prove that the sum is divisible by 7.

103. Solve the following alphamatic given that different letters stand for different digits $0, 1, 2, 3, \ldots, 9$:

$$\begin{array}{r} FORTY \\ TEN \\ TEN \\ \hline SIXTY \\ \hline \end{array}$$

104. In a class of 25 students, there are 17 cyclists, 13 swimmers and 8 weight lifters and no one is all the three. In a certain mathematics examination 6 students got grades D or E. If the cyclists, swimmers and weight lifters all got grade B or C, determine the number of students who got grade A. Also find the number of cyclists who are swimmers.

105. Five men A, B, C, D, E are wearing caps of black or white colour without each knowing the colour of his cap. It is known that a man wearing a black cap always speaks the truth while a man wearing a white cap always lies. If they make the following statements, find the colour of the cap worn by each of them:

A: I see three black and one white cap.

B: I see four white caps.

C: I see one black and three white caps.

D: I see four black caps.

106. Let f be a bijective (one-one and onto) function from the set $A = \{1, 2, 3, \ldots, n\}$ to itself. Show that there is a positive integer $M > 1$ such that

$$f^M(i) = f(i) \quad \text{for each} \quad i \in A.$$

[f^M denotes the composite function $f \circ f \circ \cdots \circ f$ repeated M times.]

107. Show that there exists a convex hexagon in the plane such that:

a. all its interior angles are equal,

b. its sides are $1, 2, 3, 4, 5, 6$ in some order.

108. There are ten objects with total weight 20, each of the weights being a positive integer. Given that none of the weights exceed 10, prove that the ten objects can be divided into two groups that balance each other when placed on the two pans of a balance.

109. At each of the eight corners of a cube write $+1$ or -1 arbitrarily. Then, on each of the six faces of the cube write the product of the numbers written at the four corners of that face. Add all the fourteen numbers so written down. Is it possible to arrange the numbers $+1$ and -1 at the corners initially so that this final sum is zero ?

110. Given the 7-element set $A = \{a, b, c, d, e, f, g\}$, find a collection T of 3-element subsets of A such that each pair of elements from A occurs exactly in one of the subsets of T.

Chapter 2

Toolkit

2.1 Number Theory

1. Divisibility Tests

   a. A number is divisible by 4 if and only if the two-digit number formed by the last two digits is divisible by 4. For example 4 divides 2134824 since 4 divides 24 while 4 does not divide 57892382 as 4 does not divide 82.

   b. A number is divisible by 3 (respectively 9) if and only if the sum of the digits of the number is divisible by 3 (respectively 9).

   c. A number is divisible by 11 if and only if 11 divides the difference between the sum of the 1st, 3rd, 5th, ... digits and the sum of the 2nd, 4th, 6th, ... digits. For instance, 11 divides a 4-digit number $abcd$ if and only if 11 divides $(a + c) - (b + d)$.

2. The square of any integer is either divisible by 4 or leaves remainder 1 when divided by 4. Thus, an integer which leaves a remainder 2 or 3 when divided by 4 can never be a square. If a prime p divides a square number then p^2 also divides that number.

3. Two integers are said to be relatively prime (to each other) if they have no common divisors except 1 ie., if their G. C. D is 1 . For example, 26 and 47 are relatively prime. In notation we write $(a, b) = 1$, if a and b are relatively prime. (More generally $a, b)$ denotes the G. C. D of a and b. If a divides bc and a and b are relatively prime to each other then a divides c.

4. If x is any real number then $[x]$ denotes the largest integer less than or equal to x. Thus $[\pi] = 3, \left[\frac{1}{2}\right] = 0$. We always have $(x-1) < [x] \leq x$.

5. For any integer $n, d(n)$ denotes the number of divisors of n. For example $d(4) = 3, d(5) = 2$, etc. If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ is the prime factorisation of n then $d(n) = (a_1+1)(a_2+1)\cdots(a_r+1)$.

6. i. If a and b leave the same remainder when divided by m then m divides $(a-b)$.

   ii. If the sum of the remainders of a and b when divided by m is divisible by m then $(a+b)$ is divisible by m and conversely.

7. If p is a prime number then the largest power of p dividing $n!$ is p^r where r is given by

$$r = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots.$$

   Note that this is in fact a finite sum.

8. Let $\binom{n}{r}$ denote the number of combinations of n distinct taken r at a time. Then

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

   For a prime p, we have that p divides $\binom{p}{r}$ for all r satisfying $1 \leq r \leq p-1$.

9. For a prime number p and any integer a, we have that p always divides $(a^p - a)$ This is called the Fermat's little theorem. (Hint for the proof: Induction on a and use 8 above.)

10. If a and b are integers with $(a, b) = 1$ and the product ab is and n-th power then a and b are themselves n-th powers.

2.2 Algebra

Polynomials

1. Remainder theorem: The remainder after dividing a polynomial $p(x)$ by $(x - a)$ is $p(a)$.

2. Factorization of a polynomial: If α is a zero of a polynomial $p(x)$, then $(x - \alpha)$ is a factor of $p(x)$. This follows from (1) above since $p(\alpha) = 0$. If $\alpha_1, \alpha_2, \ldots \alpha_n$ are the zeros of an nth degree polynomial, then

$$p(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$$

where a is the leading coefficient of $p(x)$.

3. Fundamental theorem of Algebra: If $p(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients, then there exists a complex number β such that $p(\beta) = 0$. It follows that such a polynomial can be totally factorized; i.e., there exist complex numbers $\beta_1, \beta_2, \beta_3, \ldots, \beta_n$ such that

$$p(x) = \alpha(x - \beta_1)(x - \beta_2)\ldots(x - \beta_n)$$

4. If $p(x)$ is a polynomial with real coefficients and if α is a zero of $p(x)$, then $\bar{\alpha}$ is also a zero of $p(x)$, where $\bar{\alpha}$ is the complex conjugate of α.

5. If $p(x)$ is a polynomial with real coefficients, then $p(x)$ can be written as the product of its linear and quadratic factors: i.e., we can find real numbers $a, \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l, \gamma_1, \ldots, \gamma_l$ such that

$$\begin{aligned} p(x) &= a(x - \alpha_1) \cdots (x - \alpha_k) \\ &\quad \times (x - \beta_1)^2 + {\gamma_1}^2) \cdots ((x - \beta_l)^2 + {\gamma_l}^2). \end{aligned}$$

6. Relation between zeros and coefficients of a polynomial: Suppose $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of the polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0, \quad (a_n \neq 0).$$

Then the following relations hold:

$$\sum_{i=1}^{n} \alpha_i = -\frac{a_n - 1}{a_n}, \quad \sum_{i<j} \alpha_i \alpha_j = \frac{a_n - 2}{a_n},$$

$$\sum_{i<j<k} \alpha_i \alpha_j \alpha_k = \frac{a_n - 3}{a_n},$$

etc.,

$$\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n = (-1)^n \frac{a_0}{a_n}.$$

Using these we can calculate the sums of powers of the roots of a polynomial. For example,

$$\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = \frac{a_{n-1}^2 - 2a_{n-2}a_n}{a_n^2}.$$

Inequalities

1. AM-GM-HM inequality (AM $\geq$ GM $\geq$ HM): If $a_1, a_2, a_3, \ldots, a_n$ are n positive real numbers, their arithmetic mean, geometric mean and harmonic mean satisfy the inequalities

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \ldots a_n)^{\frac{1}{n}}$$

and

$$(a_1 a_2 \cdots a_n)^{\frac{1}{n}} \geq \frac{n}{(1/a_1) + (1/a_2) + \cdots + (1/a_n)}$$

Equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

As a consequence, we have the following useful inequality: If $a_1, a_2, \ldots, a_n$ are n positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \geq n^2.$$

2. Cauchy-Schwarz inequality: Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two sets of n real numbers. Then the following inequality holds:

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right).$$

Here equality holds if and only if there exists a real constant λ such that $a_j = \lambda b_j$ for $1 \leq j \leq n$.

3. If $\alpha a \leq \alpha b$ and α is positive, then $a \geq b$; If α is negative then $a \leq b$. Thus while canceling on both sides of an inequality one must look at the sign of the quantity that is being canceled.

4. If $\alpha \geq 1$, then $(1+x)^\alpha \geq 1 + \alpha x$ for $x > -1$. If $0 < \alpha < 1$, then $(1+x)^\alpha \leq 1 + \alpha x$ for $x > -1$.
This is known as Bernoulli's inequality.

2.3 Geometry

1. The areas of two triangles having equal bases(heights) are in the ratio of their heights(bases).

2. If ABC and DEF are two triangles, then the following statements are equivalent:

   a) $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$.

   b) $\dfrac{BC}{EF} = \dfrac{CA}{FD} = \dfrac{AB}{DE}$.

   c) $\dfrac{AB}{AC} = \dfrac{DE}{DF}$ and $\angle A = \angle D$.

   Two triangles satisfying any one of these conditions are said to be similar to each other.

3. Appolonius Theorem: If D is the midpoint of the side BC in a triangle ABC then $AB^2 + AC^2 = 2(AD^2 + BD^2)$.

4. Ceva's Theorem: If ABC is a triangle, P is a point in its plane and AP, BP, CP meet the sides BC, CA, AB in D, E, F respectively, then
$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = +1.$$
Conversely, if D, E, F are points on the (possibly extended) sides BC, CA, AB respectively and the above relation holds good, then AD, BE, CF concur at a point.
Lines such as AD, BE, CF are called Cevians.

5. Menelaus's Theorem: If ABC is a triangle and a line meets the sides BC, CA, AB in D, E, F respectively then
$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1,$$
taking directions of the line segments into consideration i.e., for example, $CD = -DC$. Conversely if on the sides BC, CA, AB (possibly extended) of a triangle ABC, points D, E, F are taken respectively such that the above relation holds good, then D, E, F are collinear.

6. If two chords AB, CD of a circle intersect at a point O (which may lie inside or outside the circle), then $AO \cdot OB = CO \cdot OD$. Conversely, if AB and CD are two line segments intersecting at O such that $AO \cdot OB = CO \cdot OD$, then the four points A, B, C, D are concyclic.

7. (This may be considered as a limiting case of 6, in which A and B coincide and the chord AB becomes the tangent at A).

   If OA is a tangent to a circle at A from a point O outside the circle and OCD is any secant of the circle (that is, a straight line passing through O and intersecting the circle at C and D), then $OA^2 = OC \cdot OD$. Conversely, if OA and OCD are two distinct line segments such that $OA^2 = OC \cdot OD$, then OA is a tangent at A to the circumcircle of triangle ABC.

8. Ptolemy's theorem: If $ABCD$ is a cyclic quadrilateral, then

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

Conversely, if in a quadrilateral $ABCD$ the above relation is true, then the quadrilateral is cyclic.

9. If AB is a line segment in a plane, then set of points P in the plane such that $\frac{AP}{PB}$ is a fixed ratio λ ($\neq 0$ or 1) constitute a circle, called the Appolonius circle. If C and D are two points on AB dividing the line segment AB in the ratio $\lambda : 1$ internally and externally, then C and D themselves are two points on the circle such that CD is a diameter. Further for any point P on the circle, PC and PD are the internal and external bisectors of $\angle APB$.

10. Two plane figures α and β such as triangles, circles, arcs of a circle are said to be homothetic relative to a point O (in the plane) if for every point A on α, OA meets β in a point B such that $\frac{OA}{OB}$ is a fixed ratio λ ($\neq 0$). The point O is called the centre of similitude or homethety. Also any two corresponding points X and Y of the figures α and β (e.g., the circumcentres of two homothetic triangles) are such that O, X, Y are collinear and $\frac{OX}{OY} = \lambda$.

2.4 Trigonometry

A. Compound and Multiple Angles.

(i) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$;

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$;

$\tan(A \pm B) = (\tan A \pm \tan B)/(1 \mp \tan A \tan B)$.

(ii) $\sin 2A = 2 \sin A \cos A = (2 \tan A)/(1 + \tan^2 A)$;

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A = (1 - \tan^2 A)/(1 + \tan^2 A);$$

$\tan 2A = (2 \tan A)(1 - \tan^2 A)$.

(iii) $\sin 3A = 3\sin A - 4\sin^3 A;$

$\cos 3A = 4\cos^3 A - 3\cos A;$

$\tan 3A = (3\tan A - \tan^3 A)/(1 - 3\tan^2 A).$

B. Conversion Formulae.

$$\sin C + \sin D = 2\sin\ [(C + D)/2];$$

$$\sin C - \sin D = 2\sin\ [(C - D)/2];$$

$$\cos C + \cos D = 2\cos\ [(C + D)/2]\ \cos\ [(C - D)/2];$$

$$\cos C - \cos D = 2\sin\ [(C + D)/2]\ \sin\ [(D - C)/2];$$

$$2\sin A\cos B = \sin(A + B) + \sin(A - B);$$

$$2\cos A\sin B = \sin(A + B) - \sin(A - B);$$

$$2\cos A\cos B = \cos(A + B) + \cos(A - B);$$

$$2\sin A\sin B = \cos(A - B) - \cos(A + B).$$

C. Properties of triangles.

Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R;$

Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A;$

Half angle rule:

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \qquad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}};$$

Circumradius: $R = \dfrac{abc}{4(\text{Area})};$

In-radius: $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = (s-a)\tan\dfrac{A}{2}$

Area: $\Delta = rs = \frac{1}{2}bc\sin A = 2R^2\sin A\sin B\sin C$

$$= \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)};$$

Medians: $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$ and similar expressions for other medians.

D. Miscellaneous.

(a) $a = b\cos C + c\cos B$, $b = a\cos C + c\cos A$, $c = a\cos B + b\cos A$.

(b) If O is the circumcentre and X is the mid-point of BC, then $\angle BOX = \angle COX = A$ and $OX = R\cos A$.

(c) If AD is the altitude with D on BC and H the orthocentre, then $AH = 2R\cos A$, $HD = 2R\cos B\cos C$.

(d) If G is the centroid and N the nine-point centre, then O, G, N, H are collinear and $OG : GH = 1 : 2$, $ON = NH$.

(e) If I is the in-centre, then $\angle BIC = 90° + (A/2)$.

(f) The centroid divides the medians in the ratio 2 : 1.

(g) $OI^2 = R^2 - 2Rr = R^2[1 - 8\sin(A/2)\sin(B/2)\sin(C/2)]$;
$OH^2 = R^2(1 - 8\cos A\cos B\cos C) = 9R^2 - a^2 - b^2 - c^2$;
$HI^2 = 2r^2 - 4R^2\cos A\cos B\cos C$.

(h) If $A + B + C = \pi$, then
$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$,
$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$,
$\tan A + \tan B + \tan C = \tan A\tan B\tan C$,
$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$,
$\cos^2 A + \cos^2 B + \cos^2 C = 1 + 2\cos A\cos B\cos C$.

(i) Area of a quadrilateral $ABCD$ with $AB = a$, $BC = b$, $CD = c$, $DA = d$, $A + C = 2\alpha$ is given by

$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha}.$$

If it is cyclic, then

$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

Its diagonals are given by

$$AC = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}, \quad BD = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}.$$

2.5 Combinatorics

1. a. The sum rule: If A and B are two disjoint finite sets, then $|A \cup B| = |A| + |B|$.
   In general, If $A_1, A_2, \ldots, A_n$ are n pairwise disjoint finite sets then
   $|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$.

   b. The product rule: If A and B are two finite sets, then $|A \times B| = |A| \cdot |B|$.
   In general, if $A_1, A_2, \ldots, A_n$ are n finite sets then,
   $|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \ldots |A_n|$.

2. Pigeon-hole Principle: If $n + 1$ objects are distributed at random into n boxes, then at least one box has at least two objects. A more general form of this principle is as follows: If $nk + 1$ objects are distributed at random into n boxes, then some box has at least $k + 1$ objects.

3. Principle of Inclusion and Exclusion:. If A and B are two finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$. For three finite sets A, B, C we have $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

4. Some properties of Binomial Coefficients:

   a. $\binom{n}{r} = \binom{n}{n-r}, \quad 0 \le r \le n.$

b. $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$, $1 \leq r \leq n$.

c. The binomial theorem: If n is a positive integer, then

$$\begin{aligned}(x+y)^n &= \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r \\ &= x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + y^n.\end{aligned}$$

d. (i) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

(ii) $\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots = 2^{n-1}$.

5. a. [Combinations with repetitions]: Suppose there are n types of objects and we wish to choose k elements repetitions being allowed. (We assume here that each of n types of objects is available any number of times). Then the number of such choices is $\binom{n+k-1}{k}$. Here n may be less than, equal to or greater than k.

b. [Permutations with restricted repetitions]: Suppose there are n objects of which k_1 objects are of first kind (and identical), k_2 objects are of second kind, . . . , k_r objects are of r-th kind (so that $k_1 + k_2 + k_3 + \ldots + k_r = n$). Then the number of permutations of all the n objects is

$$\frac{n!}{k_1!k_2!\ldots k_r!}$$

This number is often denoted by $\binom{n}{k_1, k_2, \ldots k_r}$ and is called a multinomial coefficient.

c. [Permutations with unrestricted repetitions]: If there are r types of objects with unlimited supply of each type, then the number of permutations formed by n of them is r^n.

6. a. The number of all subsets of an n-element set is 2^n. The number of nonempty subsets of an n-element set is $2^n - 1$.

   b. The number of one-one functions from an m-set to an n-set ($m \leq n$) is ${}^nP_m = \frac{n!}{(n-m)} = n(n-1)(n-2)...(n-m+1)$.

   c. The number of bijections from an n-set to itself is $n!$.

   d. The number of functions from an m-set to an n-set is n^m.

   e. The number of distinct terms (monomials) in the expansion of $(x_1 + x_2 + ... + x_r)^n$ is $\binom{n+r-1}{r}$.

7. Suppose for each non-negative integer n is associated a quantity x_n. If x_n can be expressed in terms of $xn-1, xn-2, ..., x_0$ using a relation, such a relation is called a reccurence relation. For example, if F_n is the n-th Fibonacci number, then

$$F_{n+1} = F_n + Fn - 1, n \geq 1,$$

where $F_0 = F_1 = 1$. See also problem 83.

8. An useful technique in combinatorial problems, especially identities, is counting in two ways. For example, the relation

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

shows that the number of combination of n objects taken r at a time can be obtained by counting it in another way as follows. We fix an object, and from among the remaining $n-1$ objects we count the number of combinations of r objects to get $\binom{n-1}{r}$. Thus we have left out the possibility of this fixed object to be one among the selected. The number of such possibilities is precisely equal to $\binom{n-1}{r-1}$. See also problem 78.

Chapter 3

Solutions

3.1 Number Theory

1. Let the required number be $...abc7$. Since it is given that

$$5(...abc7) = 7...abc$$

we find that $c = 5$. Putting this value of c back in the equation we have $5(...ab57) = 7...ab5$ we gives $b = 8$. Continuing this way till we get 7 for the first time, we find that the required number is 142857.

2. To test the divisibility of the number $N = 19202122...919293$ by 3 or 9 we should find the sum of the digits of N. Noting that 1 occurs 9 times in the digits from 19 to 93 (in 19, 21, 31,..., 91), 2 occurs 18 times (in 20, 21, 22, ..., 29, 32, 42, ..., 92) etc. we find the sum of the digits of N to be 717. This number is divisible by 3 (since $7 + 1 + 7 = 15$ is so) but not by 9. Thus the highest power of 3 dividing N is 3.

3. Clearly $Z > X, Y$. Suppose $Y \geq X$. Since $Z > Y$, we have $Z \geq Y + 1$. Now

$$\begin{aligned} X^n &= Z^n - Y^n \\ &= (Z - Y)(Z^{n-1} + Z^{n-2}Y + ... + Y^{n-1}) \end{aligned}$$

$$> \quad ((Y+1)-Y)nX^{n-1}$$

i.e.

$$X^n > nX^{n-1} \quad \text{or} \quad X > n.$$

Thus Z, X, Y are all bigger than n.

4. Suppose, on the contrary, that

$$\frac{\log_{10} m}{\log_{10} n} = \frac{a}{b}$$

where a and b are integers. But then we would have $m^b = n^a$ which is impossible since m and n have no common factors and a and b are integers. Therefore the given number is not rational.

5. First note that the set of primes dividing x is the same as the set of primes dividing y. Take any prime p dividing x (and hence y also) and suppose it occur to the power α in x and β in y (that is, p^α is the maximum power of p dividing x and p^β is the maximum power of p dividing y). Then

$$x^a = y^b \Rightarrow p^{\alpha a} = p^{\beta b} \Rightarrow \alpha a = \beta b$$

$$\Rightarrow a \mid \beta b \text{ and } b \mid \alpha a$$

$$\Rightarrow a \mid \beta \text{ and } b \mid \alpha \text{ since } (a, b) = 1.$$

Write $\beta = a\beta_p$ and $\alpha = b\alpha_p$. Then

$$p^{\alpha a} = p^{\beta b} \Rightarrow p^{ab\alpha_p} = p^{ab\beta_p}$$

$$\Rightarrow \alpha_p = \beta_p.$$

For each prime p dividing x (and hence y) get the integer α_p. Verify that the integer $n = \prod_{p|n} p^{\alpha_p}$ (this notation means n is the product of the numbers p^{α_p} for each prime p dividing n) satisfies the required properties.

6. Suppose $2^m + 3^n = a^2$. Since any square number will leave remainder 0 or 1 when divided by 3 we get that m is an even number (as any odd power of 2 leaves remainder 2 when divided by 3). Similarly, using the fact that any square number is either divisible by 4 or will leave remainder 1 when divided by 4 we find that n is also an even number. Put $m = 2r$ and $n = 2s$. We have

$$2^{2r} = a^2 - 3^{2s} = (a - 3^s)(a + 3^s).$$

Hence

$$(a - 3^s) = 2^i \quad \text{and} \quad (a + 3^s) = 2^{2r-i}.$$

We would then have $2 \cdot 3^s = 2^i(2^{2r-2i} - 1)$, which implies that $i = 1$. Thus $a - 3^s = 2$ and $a + 3^s = 2^{2r-1}$ i.e., $3^s = 2^{2r-2} - 1$. Suppose $s > 1$. Then $r \geq 3$. But then the above equation is impossible since when divided by 8, the left hand side 3^s would leave remainder 1 or 3 while the right hand side would leave the remainder 7. Thus $s = 1$ is the only possibility; when $s = 1$, that is $n = 2$, we have the solution $2^4 + 3^2 = 25$. Thus $(m, n) = (4, 2)$ is the only solution.

7. If n is even, then $n^4 + 4^n$ is also an even number greater the 2 and hence not a prime. So let n be odd; we will show that $n^4 + 4^n$ can always be factored: (Note that n is odd $\Rightarrow n + 1$ is even.)

$$n^4 + 4^n = n^4 + 2^{2n} = (n^2 + 2^n)^2 - 2^{n+1}n^2$$

$$[(n^2 + 2^n) + 2^{(n+1)/2}n][(n^2 + 2^n) - 2^{(n+1)/2}n].$$

It is only required to observe that each of the factors above is greater than 1 when $n > 1$.

8. Let $n = aabb$ be a number satisfying the given properties. Since n is a square the only possibilities for b are 1,4,5,6 or 9. Among them 1,5,6 and 9 are not possible since the numbers $aa11, aa55$ and $aa99$ leave remainder 3 and $aa66$ leaves remainder 2 when divided by 4, which is not possible if n is a square. So b can only be 4. Clearly 11 divides $n = aabb$. Since n is a square and 11

is a prime, 11^2 also divides $n = 11 \times a0b$ that is, 11 divides $a0b$ which implies 11 divides $a + b$. Since b can be only 4, the only possibility for a is $a = 7$. Noting that $7744 = (88)^2$ is indeed a square, we conclude that 7744 is the only number with the given properties.

9. This problem uses the fact that the cube of any integer when divided by 7 leaves remainder 0, 1 or 6. If any of the numbers a, b, c is divisible by 7, then the given expression is also divisible by 7. So suppose none of a, b, c is divisible by 7. Then the possible remainders for a^3, b^3, c^3 when divided by 7 are 1 and 6. Since there are three numbers, a^3, b^3, c^3, and only two possible remainders, 1 or 6, at least two numbers, say, a^3 and b^3, leave the same remainder. But then $(a^3 - b^3)$ and hence the given expression is divisible by 7.

10. We have $\binom{2000}{1000} = \frac{2000!}{(1000!)^2}$. We should look for a 3-digit prime which occurs more often in the numerator than in the denominator so that it survives in $\binom{2000}{1000}$; (since the denominator is a square, primes always occur to even power in the denominator) that is, we should look for a prime which occurs only twice in the denominator but thrice in the numerator. Any prime in the range from 5000 to $2000/3 = 666\frac{2}{3}$ will satisfy our requirement and 661 is the largest prime in that range.

11. We give two solutions:

Solution (i): Since a, b, c are all positive integers, $a > c$ and $b > c$; say, $a = c + m$ and $b = c + n$ for some (positive) integers m and n. Then we have

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c} \Rightarrow \frac{1}{c+m} + \frac{1}{c+n} = \frac{1}{c}.$$

On simplification we get $c^2 = mn$.

Here m and n cannot have any common factors. For, if $d > 1$ divides m and n then it divides c and so a and b as well, which is not possible since it is given that a, b and c have no common factors (other than 1). Then m and n are themselves squares, say $m = k^2$ and $n = l^2$. Then

$$a + b = c + m + c + n = m + n + 2c = k^2 + l^2 + 2kl = (k + l)^2.$$

Solution (ii): Let $d = (a, b)$ be the $g.c.d.$ of a and b.We will show that $a + b = d^2$. Write $a = a_1 d$ and $b = b_1 d$. Since d is the $g.c.d$ of a and b, $(a_1, b_1) = 1$. We thus have

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{d}\left(\frac{1}{a_1} + \frac{1}{b_1}\right) = \frac{1}{c}$$

or

$$\frac{a_1 + b_1}{a_1 b_1} = \frac{d}{c}.$$

Observing that $(d, c) = 1$ and $(a_1 + b_1, a_1 b_1) = 1$, we get that $a_1 + b_1 = d$ and $a_1 b_1 = c$. But then

$$a + b = d(a_1 + b_1) = d^2.$$

12. We are to find an n such that 10^{1993} is the highest power of 10 dividing $n!$. Since multiples of 2 occur far more often than multiples of 5 it is enough to find an n such that 5^{1993} is the maximum power of 5 that divides $n!$, that is we have to solve for n in the equation:

$$1993 = \left[\frac{n}{5}\right] + \left[\frac{n}{25}\right] + \left[\frac{n}{125}\right] + \cdots.$$

Now

$$\begin{aligned}\left[\frac{n}{5}\right] + \left[\frac{n}{25}\right] + \left[\frac{n}{125}\right] + \cdots &\leq \frac{n}{5} + \frac{n}{25} + \frac{n}{125} + \cdots \\ &\leq \frac{n}{5}\left[1 + \frac{1}{5} + \frac{1}{25} + \cdots\right] \\ &\leq \frac{n}{5}\left(\frac{5}{4}\right) = \frac{n}{4};\end{aligned}$$

i.e., $n \geq 7972$. Since

$$\left[\frac{7972}{5}\right] + \left[\frac{7972}{25}\right] + \left[\frac{7972}{125}\right] + \left[\frac{7972}{625}\right] + \left[\frac{7972}{3125}\right] = 1989,$$

we have to arrange for four more multiples of 5. Note that 7975! will have one more multiple of 5 and one more multiple of 25 than 7972! and 7985! will have 3 more multiples of 5 and 1 more multiple of 25 (that is, in all 4 more multiples of 5) than 7972!. Thus 7985 is the required number. Note that 7986, 7987, 7988 and 7989 also satisfy the requirement.

13. Note that $1990 = 199 \times 10$ and that 199 is a prime number. By Fermat's (little) theorem 199 divides $(2^{199} - 2)$, that is,

$$2^{199} = 199n + 2,$$

for some integer n. Raising both sides to 10^{th} power we get

$$\begin{aligned} 2^{1990} &= (199n+2)^{10} \\ &= (199n)^{10} + \binom{10}{1}(199n)^9 \cdot 2 + \cdots + 2^{10}. \end{aligned}$$

Note that 2^{1990} and 2^{10} have the same last digit, 4, and thus 10 divides $2^{1990} - 2^{10}$. But 199 also divides $(2^{1990} - 2^{10})$. Therefore 1990 divides $(2^{1990} - 2^{10})$ or, in other words, $2^{10} = 1024$ is the remainder when 2^{1990} is divided by 1990.

14. We have the obvious solutions $(7, 0)$ and $(0, 7)$. So suppose $x \neq 0$ and $y \neq 0$. We have

$$(xy-7)^2 = x^2 + y^2$$

or

$$(xy)^2 - 14xy + 49 = x^2 + y^2$$

or

$$(xy)^2 - 12xy + 36 + 13 = x^2 + y^2 + 2xy$$

or

$$(xy-6)^2 + 13 = (x+y)^2$$

or

$$13 = [(x+y) + (xy-6)][(x+y) - (xy-6)].$$

Since 13 is a prime number the only possible factors are ± 1 and ± 13, i.e.,
(i) $(x + y) + (xy - 6) = 13$ and $(x + y) - (xy - 6) = 1$
or
(ii) $(x + y) - (xy - 6) = -13$ and $(x + y) + (xy - 6) = -1$. When solved, these alternatives give the solutions $(3, 4)$ and $(4, 3)$. Thus, $(7, 0), (0, 7), (3, 4)$ and $(4, 3)$ are all the solutions (in non-negative integers) of $(xy - 7)^2 = x^2 + y^2$.

15. Let $[\sqrt{n}] = k$. Then $k^2 < n < (k + 1)^2$. Also since k^3 divides n^2, we have that k^2 divides n^2 and hence k divides n. Thus the only possibilities for n are $n = k^2 + k$ and $n = k^2 + 2k$.
(i) Let $n = k^2 + k$. Then

$$\begin{aligned} k^3 \mid n^2 &\Rightarrow k^3 \mid (k^2 + k)^2 = k^4 + 2k^3 + k^2 \\ &\Rightarrow k^3 \mid k^2 \Rightarrow k = 1 \end{aligned}$$

i.e., $n = 2$.

(ii) Let $n = k^2 + 2k$. Then

$$k^3 \mid n^2 \Rightarrow k^3 \mid (k^2 + 2k)^2 = k^4 + 4k^3 + 4k^2$$

which implies that $k^3 \mid 4k^2$ or $k \mid 4$. Therefore, $k = 1, 2$ or 4. When $k = 1, 2, 4$, we get the corresponding values $3, 8$ and 24 for n. Thus $n = 2, 3, 8$ and 24 are all positive integers satisfying the given conditions.

16. Consider the product $n(n + 1)(n + 2)(n + 3)$ of four consecutive numbers. Suppose $n > 1$ as $1 \cdot 2 \cdot 3 \cdot 4 = 24$ is anyway not a cube. We use the fact that if the product of two relatively prime numbers is a cube then each of the two numbers is itself a cube.

Case (i): Suppose n is even. Then $(n + 1)$ is relatively prime to $n(n + 2)(n + 3)$. Thus if $n(n + 1)(n + 2)(n + 3)$ is a cube we must have that $(n + 1)$ and $n(n + 1)(n + 3)$ are cubes. But $n(n + 2)(n + 3)$ is not a cube since it lies strictly between two consecutive cubes:

$$(n + 1)^3 = n^3 + 3n^2 + 3n + 1 < n(n + 2)(n + 3)$$

and

$$n(n+2)(n+3) < (n+2)^3 = n^3 + 6n^2 + 12n + 8.$$

Thus, $n(n+1)(n+2)(n+3)$ is not a cube for even n.

Case(ii) Suppose n is odd. The proof is similar by noting that this time $n(n+1)(n+3)$ is prime to $(n+2)$.

17. For both parts, the proof will be by induction on n. (i) For $n = 1$, $3^2 \mid 2^3 + 1$ and so the statement is true for $n = 1$.
Suppose $3^{k+1} \mid 2^{3^k} + 1$ for some k. We have to show that $3^{k+2} \mid 2^{3^{k+1}} + 1$.

$$3^{k+1} \mid 2^{3^k} + 1 \Rightarrow (3^{k+1})^3 \mid \left(2^{3^k} + 1\right)^3$$

i.e. $3^{3k+3} \mid 2^{3^{k+1}} + 1 + 3 \cdot 2^{3^k}(2^{3^k} + 1)$.

Since $3k + 3 > k + 2$, the above expression implies that

$3^{k+2} \mid 2^{3^{k+1}} + 1 + 3.2^{3^k}(2^{3^k} + 1)$.

But $3^{k+1} \mid 2^{3^k} + 1$ by the induction hypothesis and hence

$3^{k+2} \mid 3\left(2^{3^k} + 1\right)$. Hence $3^{k+2} \mid 2^{3^{k+1}} + 1$ and we are through.

(ii) Here the induction hypothesis is that for any n, 3^{n+2} does not divide $2^{3^n} + 1$. For $n = 1$ it is true that 3^3 does not divide $2^3 + 1$. Suppose that 3^{k+2} does not divide $2^{3^k} + 1$ for some k.

Proceeding as in (i) we get that

$$3^{k+3} \mid 2^{3^{k+1}} + 1 + 3.2^{3^k}\left(2^{3^k} + 1\right).$$

Now if 3^{k+3} divides $2^{3^{k+1}} + 1$, then 3^{k+3} divides $3 \cdot 2^{3^k}(2^{3^k} + 1)$ which in turn implies that 3^{k+2} divides $2^{3^k} + 1$ which is a contradiction to the induction hypothesis. Therefore 3^{k+3} does not divide $2^{3^{k+1}} + 1$ and that finishes the proof.

18. Since x, y are positive integers we have $x > n$ and $y > n$. Write $x = n + a$ and $y = n + b$ where a, b are positive integers. Thus

$$\frac{1}{n+a} + \frac{1}{n+b} = \frac{1}{n}$$

which simplifies to $n^2 = a \cdot b$.

Retracing the steps, we find that if n^2 is written as a product of two numbers say, p and q then

$$\frac{1}{n+p} + \frac{1}{n+q} = \frac{1}{n}.$$

Thus the number of solutions of $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$ for a given n is the same as the number of divisors of n^2, $d(n^2)$. If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime factorisation of n we know that

$$d(n^2) = (2\alpha_1 + 1)(2\alpha_2 + 1)..(2\alpha_k + 1).$$

So if we want all the n with $s(n) = 5$, those n should satisfy:

$$d(n^2) = 5.$$

But this can happen if and only is n is of the from p^2 for a prime p. Thus $\{p^2 \mid p \text{ is prime}\}$ is the set of all positive integers n for which $s(n) = 5$.

19. Since the product of k consecutive integers is divisible by $k!$, $A(n)$ is an integer. We compare the highest powers of 2 dividing the numerator and denominator to determine the nature of $A(n)$.

Suppose we express n in the base 2, say,

$$n = a_l 2^l + a_{l-1} 2^{l-1} + a_{l-2} 2^{l-2} + \ldots + a_1 \cdot 2 + a_0, \quad a_l = 1.$$

The highest power of 2 dividing $n!$ is given by

$$s = \left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right] + \left[\frac{n}{2^3}\right] + \ldots + \left[\frac{n}{2^l}\right]$$

where $[x]$ denotes the largest integer smaller than x.

But, for $1 \le m \le l$,

$$\left[\frac{n}{2^m}\right] = a_l 2^{l-m} + a_{l-1} 2^{l-m-1} + \ldots + a_m.$$

Thus

$$\begin{aligned} s &= \sum_{m=1}^{l} \left[\frac{n}{2^m}\right] \\ &= \sum_{m=1}^{l}\sum_{k=m}^{l} a_k 2^{k-m} \\ &= \sum_{k=1}^{l} a_k \left(\sum_{m=1}^{k} 2^m - 1\right) \\ &= \sum_{k=1}^{l} a_k(2^k - 1) \\ &= \sum_{k=0}^{l} a_k 2^k - \sum_{k=0}^{l} a_k \\ &= n - \{\text{sum of the digits of } n \text{ in the base } 2\} \end{aligned}$$

Hence the highest power of 2 dividing $(n!)^2$ is $2s = 2n - 2$(sum of the digits of n in the base 2). Similarly the highest power of 2 dividing $(2n)!$ is $t = 2n-$ (sum of the digits of $2n$ in the base 2). But the digits of n in base 2 and those of $2n$ in base 2 are the same except for a zero at the end of the representation for $2n$.

Thus

$$t - 2s = a_l + a_{l-1} + a_{l-2} + \ldots + a_1 + a_0$$

where the a_i are the digits of n in base 2. Note that $a_l = 1$. Hence $t - 2s \geq 1$. But $A(n)$ is even if and only if $t - 2s \geq 1$. Hence it follows that $A(n)$ is even for all n.

Moreover $A(n)$ is divisible by 4 if and only if $t - 2s \geq 2$. Since $A(1) = 2$, 4 does not divide $A(1)$. Suppose $n = 2^l$ for some l. Then $a_l = 1$ and $a_i = 0$ for $0 \leq i \leq l - 1$. Hence $t - 2s = 1$ and $A(n)$ is not divisible by 4. On the other hand if n is not a power of 2, then for some l,

$$n = 2^l + a_{l-1}2^{l-1} + a_{l-2}2^{l-2} + \ldots + a_0$$

where $a_i \neq 0$ for at least one i and hence must be equal to 1. Thus $t - 2s \geq 1 + a_i \geq 2$. It follows that $A(n)$ is divisible by 4 if and only if n is not a power of 2.

Remark: Given any prime p, the highest power s of p dividing $n!$ is given by

$$s = \frac{n - (\text{sum of the digits of n in the base } p)}{p - 1}$$

20. First we observe that 2,3,5 divide A. So we may take $A = 2^\alpha 3^\beta 5^\gamma$. Considering $2A, 3A$ and $5A$, we observe that $\alpha + 1, \beta, \gamma$ are divisible by 2; $\alpha, \beta + 1, \gamma$ are divisible by 3; and $\alpha, \beta, \gamma + 1$ are divisible by 5. We can choose $\alpha = 15 + 30n$; $\beta = 20 + 30n$; $\gamma = 24 + 30n$. As n varies over the set of natural numbers, we get an infinite set of numbers of required type. We may also take $A = 2^{15}3^{20}5^{24}n^{30}$ to get a different such infinite set.

21. We may assume that both x, y to be positive. We observe that p is odd. Taking the difference, we get

$$p^2 - p = 2(y^2 - x^2).$$

Write this in the form $p(p - 1) = 2(y - x)(y + x)$. Since p is odd p cannot divide 2. If p divides $y - x$, then $p \leq y - x$. But then $p - 1 \geq 2y + 2x$ is not possible. Hence p should divide $y + x$. This gives $p \leq y + x$ and $p - 1 \geq 2(y - x)$. Eliminating y, we obtain $p + 1 \leq 4x$. Since $p + 1 = 2x^2$, we get $2x^2 \leq 4x$ and hence $x \leq 2$. Taking $x = 1$, we get $p = 1$, which is not a prime. If $x = 2$, we get $p = 7$. Thus $p = 7$ is the only prime satisfying the given condition.

22. We may assume that $a \geq b \geq c$. The given equation leads to $3 \leq \left(1 + \frac{1}{c}\right)^3$. If $c \geq 3$, then

$$\left(1 + \frac{1}{c}\right)^3 \leq \left(1 + \frac{1}{3}\right)^3 = \frac{64}{27} < 3.$$

We conclude that $c = 1$ or 2. Consider the case $c = 1$. We obtain

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right) = \frac{3}{2}.$$

This simplifies to $ab - 2a - 2b - 2 = 0$. This can be written in the form $(a-2)(b-2) = 6$. We get $(a-2, b-2) = (6,1)$ or $(3,2)$; i.e., $(a,b) = (8,3)$ or $(5,4)$.

In the case $c = 2$, we similarly obtain $(a-1)(b-1) = 2$. Solving this equation in positive integers, we get $(a,b) = (3,2)$. Hence the ordered solutions are $(a,b,c) = (8,3,1), (5,4,1), (3,2,2)$.

23. Take $a = 2^n + 1/2$, $b = 2^{n+1} + 1/2$. In the binomial expansions of a^k and b^k, $1 \le k \le n$, we see that all the terms except the last are integral and the last terms are each equal to $1/2^k$. Hence $a^k - b^k$ is an integer for $1 \le k \le n$.

24. For $p = 2$ the given quotient is not even an integer and so we can assume p is an odd prime. Then by Fermat's Little theorem $2^{p-1} - 1$ is divisible by p. Suppose that for some integer a, $2^{p-1} - 1 = pa^2$. Since p is odd we have that

$$(2^{(p-1)/2} - 1)(2^{(p-1)/2} + 1) = pa^2.$$

Both the factors on the L.H.S. are odd and hence they are relatively prime to each other which implies that p divides exactly one of the two factors. Therefore either

$$2^{(p-1)/2} - 1 = px^2, \quad 2^{(p-1)/2} + 1 = y^2$$

or

$$2^{(p-1)/2} - 1 = x^2, \quad 2^{(p-1)/2} + 1 = py^2.$$

Case (i): Suppose $2^{(p-1)/2} - 1 = px^2, 2^{(p-1)/2} + 1 = y^2$. But then $2^{(p-1)/2} = (y-1)(y+1)$ for which the only solution is $y = 3$ i.e., $p = 7$. When $p = 7$ observe that the given quotient is indeed a square.

Case (ii): Suppose $2^{(p-1)/2} - 1 = x^2, 2^{(p-1)/2} + 1 = py^2$. Then if

$p > 3$ we get that x^2 leaves remainder 3 when divided by 4 which is impossible. Thus p can only be equal to 3; when $p = 3$ also we observe that the given quotient is indeed a square.

Thus the primes p for which the quotient $(2^{p-1} - 1)/p$ is a square are $p = 3, 7$.

25. Eliminating z from the given set of equations, we get

$$x^3 + y^3 + \{1 - (x + y)\}^2 = 1.$$

This factors to

$$(x + y)(x^2 - xy + y^2 + x + y - 2) = 0.$$

Case 1. Suppose $x + y = 0$. Then $z = 1$ and $(x, y, z) = (m, -m, 1)$, where m is an integer give one family of solutions.

Case 2. Suppose $x + y \neq 0$. Then we must have

$$x^2 - xy + y^2 + x + y - 2 = 0.$$

This can be written in the form

$$(2x - y + 1)^2 + 3(y + 1)^2 = 12.$$

Here there are two possibilities:

$$2x - y + 1 = 0, \quad y + 1 = \pm 2;$$

or

$$2x - y + 1 = \pm 3, \quad y + 1 = \pm 1.$$

Analysing all these cases we get the following solutions:

$$(0, 1, 0), (-2, -3, 6), (1, 0, 0),$$

$$(0, -2, 3), (-2, 0, 3), (-3, -2, 6).$$

3.2 Algebra

26. The largest number is $\sqrt[3]{3}$. Obviously $1 < \sqrt[3]{3}$ and $\sqrt[2]{2} < \sqrt[3]{3}$. We show that $\sqrt[n]{n} > \sqrt[n+1]{n+1}$ for $n \geq 3$. This is equivalent to

$$\left(1 + \frac{1}{n}\right)^n < n \quad \text{for} \quad n \geq 3.$$

The binomial theorem gives

$$\left(1 + \frac{1}{n}\right)^n = 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{k}\frac{1}{n^k} + \cdots + \frac{1}{n^n}.$$

But

$$\begin{aligned}\binom{n}{k}\frac{1}{n^k} &= \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!n^k} \\ &= \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) \\ &< \frac{1}{k!}.\end{aligned}$$

Hence for $n \geq 3$

$$\begin{aligned}\left(1 + \frac{1}{n}\right)^n &< 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} \\ &< 1 + 1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}} \\ &= 1 + \frac{\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} < 1 + 2 = 3.\end{aligned}$$

Thus for $n \geq 3$,

$$\left(1 + \frac{1}{n}\right)^n < 3 \leq n.$$

27. Suppose p/q (with $(p, q) = 1$) is a rational root of the given equation

$$ax^2 + bx + c = 0.$$

Then

$$ap^2 + bpq + cq^2 = 0.$$

This relation shows that q divides a and p divides c. Since a, b, c are given to be odd it follows that so are p, q. But then in the above sum the L.H.S. is a sum of three odd numbers and hence cannot be equal to 0.

28. Observe that

$$\begin{aligned}\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 &= a^2+\frac{1}{a^2}+b^2+\frac{1}{b^2}+4\\ &= (a+b)^2-2ab+\left(\frac{1}{a}+\frac{1}{b}\right)^2\\ &\quad -\frac{2}{ab}+4\\ &= 1-2ab+\frac{1-2ab}{a^2b^2}+4.\end{aligned}$$

Using AM-GM inequality, we get

$$ab \le \left(\frac{a+b}{2}\right)^2 = \frac{1}{4}.$$

Hence we get

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \ge \left(1-\frac{1}{2}\right)(1+16)+4 = \frac{25}{2}.$$

Remark: If $a_1, a_2 \ldots, a_n$ are n positive real numbers such that $a_1 + a_2 + a_3 + \cdots + a_n = 1$, then

$$\sum_{j=1}^{n}\left(a_j+\frac{1}{a_j}\right)^2 \ge \frac{(1+n^2)^2}{n}.$$

In fact, Cauchy-Schwarz inequality gives

$$\sum_{j=1}^{n}\left(a_j+\frac{1}{a_j}\right)^2 \ge \frac{1}{n}\left[\sum_{j=1}^{n}\left(a_j+\frac{1}{a_j}\right)\right]^2.$$

But AM-HM inequality gives

$$\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq \frac{n^2}{a_1 + a_2 + \cdots + a_n} = n^2.$$

Thus

$$\sum_{j=1}^{n}\left(a_j + \frac{1}{a_j}\right)^2 \geq \frac{1}{n}(1 + n^2)^2.$$

29. Suppose there are distinct natural numbers a, b, c, d such that

$$a^3 + b^3 = c^3 + d^3$$

$$a + b = c + d.$$

The first relation gives

$$(a + b)(a^2 - ab + b^2) = (c + d)(c^2 - cd + d^2)$$

Using this second relation (note that $a + b \neq 0$) we get

$$a^2 - ab + b^2 = c^2 - cd + d^2.$$

This forces $ab = cd$. But then $a - b = \pm(c - d)$. If $a - b = c - d$, then $a = c$ and $b = d$. If $a - b = -(c - d)$, then $a = d$ and $b = c$. This contradicts the distinctness of a, b, c and d. (Observe that, the conclusion is true even when a, b, c, d are distinct real numbers such that $a + b \neq 0$.)

30. Taking $x = 1$ in the given equation:

$$(1 + x + x^2)^{25} = a_0 + a_1x + a_2x^2 + \cdots + a_{50}x^{50}$$

we get

$$3^{25} = a_0 + a_1 + a_2 + \cdots + a_{50}.$$

Similarly, $x = -1$ gives

$$1 = a_0 - a_1 + a_2 - a_3 + \cdots + a_{50}.$$

Adding these, we have

$$1 + 3^{25} = 2(a_0 + a_2 + a_4 + \cdots + a_{50}.)$$

But

$$\begin{aligned} 1 + 3^{25} &= 3^{25} - 1 + 2 \\ &= 2(3^{24} + 3^{23} + 3^{22} + \cdots + 1 + 1). \end{aligned}$$

There are even number of odd terms in the braces, and hence the sum is even. This implies that $a_0 + a_2 + a_4 + \cdots + a_{50}$ is even.

31. Assume, if possible,

$$f(x) = (x + a)(x^3 + ax^2 + bx + c).$$

Comparing the coefficients of like powers of x, we get

$$a + b = 26,$$

$$ab + c = 52,$$

$$ac + d = 78,$$

$$ad = 1989.$$

But $1989 = 3^2 \cdot 13 \cdot 17$. Thus 13 divides ad and hence 13 divides a or d but not both. If 13 divides a then 13 divides $d = 78 - ac$ which is not possible. Suppose 13 divides d. Then 13 divides ac. But since 13 does not divide a, 13 divides c which implies 13 divides $ab = 52 - c$ and so b is divisible by 13 which in turn implies 13 divides $a = 26 - b$, a contradiction. Therefore $f(x)$ has no linear factors.

If $f(x) = (x^2 + ax + b)(x^2 + cx + d)$, then again,

$$a = c = 26,$$

$$b + ac + d = 52,$$

$$ad + bc = 78,$$

$$bd = 1989.$$

Since $1989 = 3^2 \cdot 13 \cdot 17, 13$ divides bd. This implies that 13 divides b or d but not both. If 13 divides b, the 13 divides ad $(= 78 - bc)$ and hence 13 divides a. But then 13 divides d $(= 52 - b - ac)$ a contradiction. Similar argument shows that 13 divides d is also not possible. We conclude that $f(x)$ cannot be written as a product of two polynomials with integral coefficients, each of degree < 4.

32. Let $x_1, x_2, x_3, \ldots, x_6$ be the roots of the polynomial equation

$$x^6 + ax^3 + bx^2 + cx + d = 0.$$

Since a, b, c, d are not all zero, at least one x_i must be nonzero. Using the relations between the coefficients and the zeroes of a polynomial, we have

$$\sum_{j=1}^{6} x_j = 0, \sum_{1 \le i < j \le 6} x_i x_j = 0.$$

But then

$$\sum_{j=1}^{6} x_j^2 = \left(\sum_{j=1}^{6} x_i \right)^2 - 2 \sum_{1 \le i < j \le 6} x_i x_j = 0.$$

If all x_j are real, then the above relation forces $x_j = 0$ for $1 \le j \le 6$. But not all x_j are zero. We conclude that not all the roots can be real.

33. Suppose x_1, x_2, x_3 and x_4 are the roots of the equation

$$x^4 + px^3 + qx^2 + rx + s = 0.$$

Then, we have

$$x_1 + x_2 + x_3 + x_4 = -p,$$

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = q,$$

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = -r,$$

$$x_1x_2x_3x_4 = s.$$

Hence using $AM \geq HM$,

$$\begin{aligned} pr &= \left(\sum_1^4 x_i\right)\left(\sum_1^4 \frac{1}{x_i}\right)x_1x_2x_3x_4 \\ &\geq 16s. \end{aligned}$$

Similarly, using $AM \geq GM$

$$\begin{aligned} q &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \\ &\geq 6(x_1^3x_2^3x_3^3x_4^3)^{\frac{1}{6}} \\ &= 6s^{1/2}. \end{aligned}$$

This gives $q^2 \geq 36s$. Note that equality holds in both the inequalities if and only if $x_1 = x_2 = x_3 = x_4$.

34. Putting $x = 1 - a, y = 1 - b$ and $z = 1 - c$, the given inequality is equivalent to

$$\frac{1-x}{x} \cdot \frac{1-y}{y} \cdot \frac{1-z}{z} \geq 8,$$

subject to the condition

$$0 < x < 1, \quad 0 < y < 1, \quad 0 < z < 1, \quad x + y + z = 1.$$

This can be rewritten in the form

$$(1 - x)(1 - y)(1 - z) \geq 8xyz.$$

Expanding the left hand side and using $x + y + z = 1$, the given inequality reduces to

$$xy + yz + zx \geq 9xyz$$

or

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 9.$$

But this (and hence the given inequality) is a consequence of AM - HM inequality. The equality holds in the given inequality if and only if $a = b = c$.

35. Consider 2001 numbers

$$\frac{1}{k}, \quad 1001 \leq k \leq 3001.$$

Using AM - HM inequality, we get

$$\left(\sum_{k=1001}^{3001} k\right)\left(\sum_{k=1001}^{3001} \frac{1}{k}\right) > (2001)^2.$$

But

$$\sum_{k=1001}^{3001} k = (2001)^2.$$

Hence we get the inequality

$$\sum_{k=1001}^{3001} \frac{1}{k} > 1.$$

On the other hand, grouping 500 terms at a time, we also have

$$\begin{aligned} S &= \sum_{k=1001}^{3001} \frac{1}{k} \\ &< \frac{500}{1000} + \frac{500}{1500} + \frac{500}{2000} + \frac{500}{2500} + \frac{1}{3001} \\ &< \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{3000} \\ &= \frac{3851}{3000} < \frac{4}{3}. \end{aligned}$$

Note: We can sharpen the above inequality. Consider the sum

$$S = \sum_{k=n+1}^{3n+1} \frac{1}{k}$$

There are $2n+1$ terms in the sum and the middle term is $\frac{1}{2n+1}$. We can write the sum in the form

$$S = \frac{1}{2n+1} + \sum_{k=1}^{n}\left(\frac{1}{2n+1+k} + \frac{1}{2n+1-k}\right)$$

$$= \frac{1}{2n+1} + \frac{2}{(2n+1)} \sum_{k=1}^{n} \frac{1}{1 - \left(\frac{k}{2n+1}\right)^2}.$$

For $0 < a < \frac{1}{2}$, we have

$$1 + a < \frac{1}{1-a} < 1 + 2a.$$

Thus we get the bounds

$$\frac{1}{2n+1} + \frac{2}{2n+1} \sum_{k=1}^{n} \left[1 + \left(\frac{k}{2n+1}\right)^2\right] < S$$

and

$$S < \frac{1}{2n+1} + \frac{2}{2n+1} \sum_{k=1}^{n} \left[1 + 2\left(\frac{k}{2n+1}\right)^2\right].$$

This on simplification gives

$$1 + \frac{2}{(2n+1)^3} \sum_{k=1}^{n} k^2 < S < 1 + \frac{4}{(2n+1)^3} \sum_{k=1}^{n} k^2.$$

Now using the identity

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

the inequality simplifies to

$$1 + \frac{n(n+1)}{3(2n+1)^2} < S < 1 + \frac{2}{3}\frac{n(n+1)}{(2n+1)^2}.$$

But for $n \geq 1$, we also have

$$\frac{2}{9} \leq \frac{n(n+1)}{(2n+1)^2} \leq \frac{1}{4}.$$

This leads to

$$\frac{29}{27} < S < \frac{7}{6}.$$

36. We have,

$$y + z = 4 - x$$
$$y^2 + z^2 = 6 - x^2.$$

From Cauchy-Schwarz inequality we get,

$$y^2 + z^2 \geq \frac{1}{2}(y + z)^2.$$

Hence,

$$6 - x^2 \geq \frac{1}{2}(4 - x)^2.$$

This simplifies to $(3x-2)(x-2) \leq 0$. Hence we have $2/3 \leq x \leq 2$.

Suppose $x = 2$. Then $y + z = 2$, $y^2 + z^2 = 2$ which has solution $y = z = 1$. (Similarly $x = 2/3$ is also possible (verify).

Since the given relations are symmetric in x, y and z, similar assertions hold for y and z also.

37. Consider the polynomial $f(x) - 2$. This vanishes at a_1, a_2, a_3, a_4 and a_5. Hence

$$f(x) - 2 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5)g(x)$$

for some polynomial $g(x)$ with integral coefficients. If $f(b) = 9$ for some integer b, then

$$7 = (b - a_1)(b - a_2)(b - a_3)(b - a_4)(b - a_5)g(b)$$

which is impossible because the integers $b - a_1, b - a_2, \ldots, b - a_5$ are all distinct and 7 cannot be factored into more than 3 distinct numbers. [Best we can do is $7 = (-7)(-1)(1)$.]

Remark: The same conclusion holds even if $f(x)$ assumes the value 2 for only 4 distinct integers.

38. Putting $y = 1/(1 - x)$, the given functional equation can be written as

$$f(x) + f(y) = 2\left(\frac{1}{x} - y\right)$$

If we set $z = 1/(1-y)$, then $x = 1/(1-z)$. Hence we also have the relations

$$f(y) + f(z) = 2\left(\frac{1}{y} - z\right)$$

and

$$f(z) + f(x) = 2\left(\frac{1}{z} - x\right).$$

Adding the first and third relations, we get

$$2(f(x) + f(y) + f(z)) = 2\left(\frac{1}{x} - x\right) - 2y + \frac{2}{z}.$$

Using the second relation, this reduces to

$$2f(x) = 2\left(\frac{1}{x} - x\right) - 2\left(y + \frac{1}{y}\right) + 2\left(z + \frac{1}{z}\right).$$

Now using

$$y + \frac{1}{y} = \frac{1}{1-x} + 1 - x, \quad z + \frac{1}{z} = \frac{x-1}{x} + \frac{x}{x-1}$$

we get

$$f(x) = \frac{x+1}{x-1}.$$

Thus $f(x) = (x+1)/(x-1)$ is the only function satisfying the given functional equation.

39. We can write

$$\begin{aligned} p(n)p(n+1) &= \left(n^2 + an + b\right)\left((n+1)^2 + a(n+1) + b\right) \\ &= n^2(n+1)^2 + a\left\{n(n+1)^2 + n^2(n+1)\right\} \\ &\quad +b\left\{n^2 + (n+1)^2\right\} + a^2 n(n+1) + b^2 + \\ &\quad ab(2n+1) \\ &= n^2(n+1)^2 + a^2n^2 + b^2 + 2an^2(n+1) + \\ &\quad 2bn(n+1) + 2nab + a^2 n + an(n+1) + \end{aligned}$$

$$\begin{aligned}
& ab + b \\
= \; & (n(n+1) + an + b)^2 + \\
& a(n(n+1) + an + b) + b \\
= \; & p(n(n+1) + an + b)
\end{aligned}$$

Aliter: If α and β are the roots of the equation $p(x) = 0$ we can write:

$$p(x) = (x - \alpha)(x - \beta).$$

Then

$$\begin{aligned}
p(n)p(n+1) &= (n-\alpha)(n-\beta)(n+1-\alpha)(n+1-\beta) \\
&= (n-\alpha)(n-\beta+1)(n-\beta)(n-\alpha+1) \\
&= \{n(n-\beta) + n - \alpha(n-\beta) - \alpha\} \\
&\quad \times \{n(n-\alpha) + n - \beta(n-\alpha) - \beta\} \\
&= (M-\alpha)(M-\beta) = p(M)
\end{aligned}$$

where

$$\begin{aligned}
M &= n^2 - n(\alpha+\beta) + \alpha\beta + n \\
&= n^2 + na + b + n.
\end{aligned}$$

Remark: Another way is to consider the quadratic equation

$$M^2 + aM + b = (n^2 + an + b)((n+1)^2 + a(n+1) + b)$$

and to show that this equation has integer roots or equivalently, that the discriminant

$$a^2 - 4[b - (n^2 + an + b)((n+1)^2 + a(n+1) + b)]$$

is a square.

40. If $a > 1$, then for any m,

$$1 + \frac{1}{a} + \frac{1}{a^2} + \cdots + \frac{1}{a^m} = \frac{1 - \frac{1}{a^{m+1}}}{1 - \frac{1}{a}} < \frac{a}{a-1}.$$

Suppose $a_1, a_2, a_3, \ldots, a_n$ are in n distinct odd natural numbers each having no prime factor larger than 5. Thus for each i either $a_i = 1$ or the only primes dividing a_i are 3 and 5. Hence, for a sufficiently large integer m

$$\begin{aligned} \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} &< \left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^m}\right) \\ &\quad \left(1 + \frac{1}{5} + \frac{1}{5^2} + \cdots + \frac{1}{5^m}\right) \\ &< \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{8} < 2. \end{aligned}$$

41. Suppose a, b, c are distinct integers such that $p(a) = b, p(b) = c$ and $p(c) = a$. Then

$$p(a) - p(b) = b - c, \quad p(b) - p(c) = c - a,$$

$$p(c) - p(a) = a - b.$$

But for any two integers $m \neq n$, $m - n$ divides $p(m) - p(n)$. Thus we get,

$$a - b | b - c, \quad b - c | c - a, \quad c - a | a - b.$$

These force $a = b = c$, a contradiction. Hence there are no integers a, b, and c such that $p(a) = b, p(b) = c$ and $p(c) = a$.

42. Observe that $\dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b}$ equals

$$(a + b + c)\left\{\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{a+c}\right\} - 3$$

which in turn equals

$$\frac{1}{2}(a + b) + (b + c) + (c + a)\left\{\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right\} - 3.$$

But by $AM \geq HM$ we have that the above quantity is

$$\geq \frac{1}{2} \cdot 9 - 3 = \frac{3}{2}.$$

Here equality holds if and only if $a = b = c$.
Suppose a, b, c are arranged such that $a \le b \le c$. Then

$$\begin{aligned}\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} &\le \frac{a}{a+c}+\frac{c}{c+a}+\frac{c}{a+b}\\ &= 1+\frac{c}{a+b}\\ &< 1+1=2\end{aligned}$$

since $c < a + b$ by the property of a triangle.

43. The relation (i) gives

$$\begin{aligned}f(1990)-90\left[\frac{f(1990)}{90}\right] &= 1990-19\left[\frac{1990}{19}\right]\\ &= 1990-1976\\ &= 14.\end{aligned}$$

Using the relation (ii),

$$\frac{1990}{90} < \frac{f(1990)}{90} < \frac{2000}{90}$$

or

$$21\frac{10}{90} < \frac{f(1990)}{90} < 22\frac{20}{90}.$$

Thus

$$\left[\frac{f(1990)}{90}\right] = 21 \text{ or } 22.$$

If

$$\left[\frac{f(1990)}{90}\right] = 21,$$

then

$$f(1990) = 14 + 90 \cdot 21 = 1904.$$

If

$$\left[\frac{1990}{90}\right] = 22,$$

then

$$f(1990) = 1994.$$

44. The given inequality reduces to

$$a^2cd + b^2cd + c^2ab + d^2ab \le 1/64.$$

We observe that $a^2cd + b^2cd + c^2ab + d^2ab = (ac + bd)(ad + bc)$. Hence using the inequality $xy \le (x + y)^2/4$, we get

$$a^2cd + b^2cd + c^2ab + d^2ab \le \frac{(ac + bd + ad + bc)^2}{4}.$$

But $ac + bd + ad + bc = (a + b)(c + d)$. One more application of the same inequality gives $ac + bd + ad + bc \le (a + b + c + d)^2/4$. Combining both these inequalities and using the data that $a + b + c + d \le 1$, we get the required inequality.

45. If $(x - \alpha)$ divides a polynomial $q(x)$ then $q(\alpha) = 0$. Let $p(x) = ax^3 + bx^2 + cx + d$. Since $(x - 1)$ divides $p(x) + 2$, we get

$$a + b + c + d + 2 = 0.$$

Hence $d = -a - b - c - 2$ and

$$\begin{aligned} p(x) + 2 &= a(x^3 - 1) + b(x^2 - 1) + c(x - 1) \\ &= (x - 1)\{a(x^2 + x + 1) + b(x + 1) + c\}. \end{aligned}$$

Since $(x - 1)^2$ divides $p(x) + 2$, we conclude that $(x - 1)$ divides $a(x^2+x+1)+b(x+1)+c$. This implies that $3a+2b+c = 0$. Similarly, using the information that $(x + 1)^2$ divides $p(x) - 2$, we get two more relations: $-a + b - c + d - 2 = 0$; $3a - 2b + c = 0$. Solving these for a, b, c, d, we obtain $b = d = 0$, and $a = 1, c = -3$. Thus there is only one polynomial satisfying the given condition: $p(x) = x^3 - 3x$.

46. Since the right hand side is positive, so is the left hand side. Hence x must be positive.

Let $x = n + f$, where $n = [x]$ and $f = \{x\}$. We consider two cases:

Case I. $0 \le f < 1/2$: In this case, we get $[2x] = [2n + 2f] = 2n$, as $2f < 1$. Hence the equation becomes

$$\frac{1}{n} + \frac{1}{2n} = f + \frac{1}{3}.$$

This forces $(1/n) + (1/2n) \geq 1/3$. We conclude that $2n - 9 \leq 0$. Thus n can take values 1,2,3,4. Among these $n = 2,3,4$ are all admissible, because for $n = 2,3,4$ we get $f = 5/12, /6, 1/24$ respectively which are all less than 1/2; while $n = 1$ is not admissible, because $n = 1$ gives $f > 1/2$. We get three solutions in this case; $x = 2 + (5/12) = 29/12$; $x = 3 + (1/6) = 19/6$; $x = 4 + (1/24) = 97/24$.

Case II. $(1/2) \leq f < 1$: Now we get $[2x] = 2n+1$, as $1 \leq 2f < 2$. The given equation reduces to

$$\frac{1}{n} + \frac{1}{2n+1} = f + \frac{1}{3}.$$

We conclude, as in **Case I**, $1/n + 1/(2n+1) \geq 1/2 + 1/3$. This reduces to $10n^2 - 13n - 6 \leq 0$. It follows that $n = 1$. But this is not admissible since $n = 1$ gives $f = 1$. We do not have any solution in this case.

47. Using the relations between zeros and coefficients of a polynomial, we obtain

$$p + q + r = 3p, \quad pq + qr + rp = 3q^2, \quad pqr = r^3.$$

The last relation shows that either $r = 0$ or $pq = r^2$. In the first case the other two relations give $q = 2p$ and $pq = 3q^2$. This would force $p(2p) = 3(2p)^2$ and hence $p = 0$. This in turn leads to $q = 0$ and we have the desired result.

If $pq = r^2$, we obtain

$$q + r = 2p, \quad r^2 + qr + rp = 3q^2.$$

Multiplying the first relation here by r and using thus obtained relation in the second , we get $3pr = 3q^2$ and hence $pr = q^2$. This with $pq = r^2$ gives $q^3 = r^3$. Thus either $q = r$ or $q^2 + qr + r^2 = 0$. In the latter case, we get $0 = q^2 + qr + r^2 = pr + qr + pq = 3q^2$ and hence $q = 0$. But then $pq = r^2$ shows that $r = 0$ and hence $p = 0$. Otherwise $q = r$ and $p + q + r = 3p$ would give $2q = 2p$ so that $r = q = p$. In all cases we obtain $p = q = r$.

48. By looking at the first few values of a_n, we guess that

$$a_n = (n-1)^2 + 1 = n^2 - 2n + 2.$$

We prove this by induction on n. In fact,

$$\begin{aligned} a_{n+1} &= 2a_n - a_{n-1} + 2 \\ &= 2[(n-1)^2+1] - [(n-2)^2+1] + 2 \\ &= 2n^2 - 4n + 4 - (n^2 - 4n + 5) + 2 \\ &= n^2 + 1. \end{aligned}$$

Now we have,

$$\begin{aligned} a_m a_{m+1} &= [(m-1)^2+1][m^2+1] \\ &= m^2(m-1)^2 + m^2 + (m-1)^2 + 1 \\ &= [m(m-1)+1]^2 + 1 \\ &= a_{m^2-m+2}. \end{aligned}$$

49. Let α, β, γ be the roots of the given cubic $x^3 - ax + b = 0$, where $a > 0$ and $b > 0$. We have then

$$\left.\begin{array}{c} \alpha+\beta+\gamma = 0 \\ \alpha\beta+\beta\gamma+\gamma\alpha = -a \\ \alpha\beta\gamma = -b. \end{array}\right\}. \qquad (*)$$

From the last of these equations, we see that either all the roots are negative or two are positive and one negative. However the second equation in (*) shows that all three cannot be negative. So two of α, β, γ are positive and the remaining root is negative. The first equation in (*) implies that the negative root is numerically larger than the other two positive roots. Hence we may assume that $\gamma < 0 < \alpha \le \beta$ where $|\alpha| \le |\beta| \le |\gamma|$.

We have

$$\begin{aligned} b - a\alpha &= -\alpha\beta\gamma + \alpha(\alpha\beta+\beta\gamma+\gamma\alpha) \\ &= \alpha^2(\beta+\gamma) = -\alpha^3 < 0. \end{aligned}$$

Since a is positive, we get $b/a < \alpha$ proving the first inequality.

Again, we have

$$\begin{aligned}
3b - 2a\alpha &= -3\alpha\beta\gamma + 2\alpha(\alpha\beta + \beta\gamma + \gamma\alpha) \\
&= -\alpha\beta\gamma + 2\alpha^2\beta + 2\alpha^2\gamma \\
&= \alpha[2\alpha(\beta + \gamma) - \beta\gamma] \\
&= \alpha[-2(\beta + \gamma)^2 - \beta\gamma] \quad (\text{since } \alpha = -(\beta + \gamma)) \\
&= -\alpha(2\beta^2 + 5\beta\gamma + 2\gamma^2) \\
&= -\alpha(2\beta + \gamma)(\beta + 2\gamma) \\
&= -\alpha(\beta - \alpha)(\gamma - \alpha).
\end{aligned}$$

Observe that $-\alpha < 0, \beta \geq \alpha, \gamma - \alpha < 0$. Hence $3b - 2a\alpha$ is nonnegative. This proves the second inequality, $\alpha \leq 3b/2a$.

50. Since λ is a root of the equation $x^3 + ax^2 + bx + c = 0$, we have

$$\lambda^3 = -a\lambda^2 - b\lambda - c.$$

This implies that

$$\begin{aligned}
\lambda^4 &= -a\lambda^3 - b\lambda^2 - c\lambda \\
&= (1 - a)\lambda^3 + (a - b)\lambda^2 + (b - c)\lambda + c
\end{aligned}$$

where we have used again

$$-\lambda^3 - a\lambda^2 - b\lambda - c = 0.$$

Suppose $|\lambda| \geq 1$. Then we obtain

$$\begin{aligned}
|\lambda|^4 &\leq (1 - a)|\lambda|^3 + (a - b)|\lambda|^2 + (b - c)|\lambda| + c \\
&\leq (1 - a)|\lambda|^3 + (a - b)|\lambda|^3 + (b - c)|\lambda|^3 + c|\lambda|^3 \\
&\leq |\lambda|^3.
\end{aligned}$$

This shows that $|\lambda| \leq 1$. Hence the only possibility in this case is $|\lambda| = 1$. We conclude that $|\lambda| \leq 1$ is always true.

3.3 Geometry

51. First assume that in the triangle ABC, $A = 2B$. Produce CA to D such that $AD = AB$. Join BD.

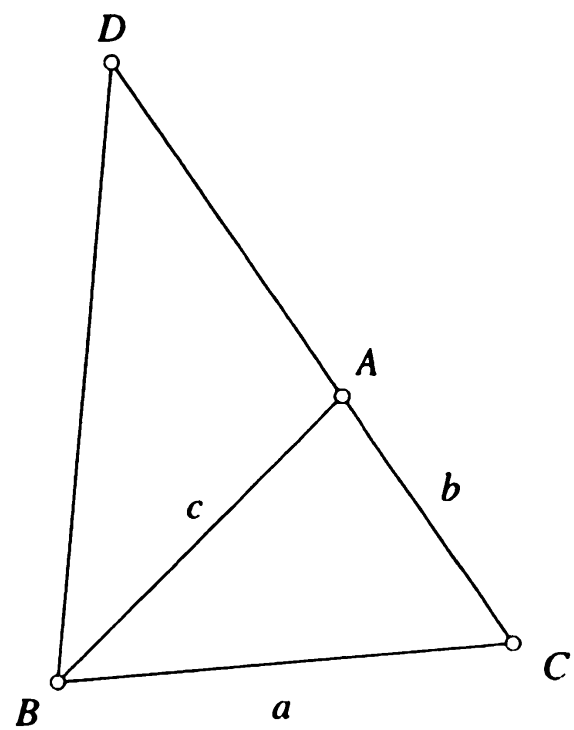

Figure 1

By construction, it is clear that ABD is an isosceles triangle and so

$$\angle ADB = \angle ABD.$$

But

$$\angle ADB + \angle ABD = \angle BAC$$

(the external angle)
Hence

$$\angle ADB = \angle ABD = \frac{A}{2} = B.$$

In triangles ABC and BDC we have $\angle ABC = \angle BDC$ and $\angle C$ is common. So $\triangle ABC$ is similar to $\triangle BDC$. Therefore,

$$\frac{AC}{BC} = \frac{BC}{DC}.$$

It follows that

$$a^2 = b(b + c).$$

Now we prove the converse. Assume that $a^2 = b(b + c)$. We refer to the same figure. As before, in the isosceles triangle ABD, we have

$$\angle ABD = \angle ADB.$$

So each of these angles is equal to half of their sum which is A. Thus, in particular,

$$\angle ADB = \frac{A}{2}. \tag{1}$$

On the other hand, in triangles ACB and BCD, we have, as a consequence of the assumption $a^2 = b(b + c)$,

$$\frac{AC}{BC} = \frac{BC}{DC},$$

and $\angle C$ is common.

So the two triangles are similar and

$$\angle CDB = \angle CBA = B. \tag{2}$$

From (1) and (2), it follows that $B = A/2$, as desired.

Aliter: We may use the Sine rule for a triangle to dispose of both the implications simultaneously.

$$\begin{aligned} A = 2B &\iff A - B = B \\ &\iff \sin(A - B) = \sin B \\ &\iff \sin(A - B)\sin(A + B) = \sin B \sin C \\ &\iff \sin^2 A - \sin^2 B = \sin B \sin C \end{aligned}$$

$$\iff (2R\sin A)^2 - (2R\sin B)^2 = (2R\sin B)(2R\sin C)$$
$$\iff a^2 - b^2 = bc$$
$$\iff a^2 = b(b+c).$$

52. Join PQ, BZ and AX (see figure 2).

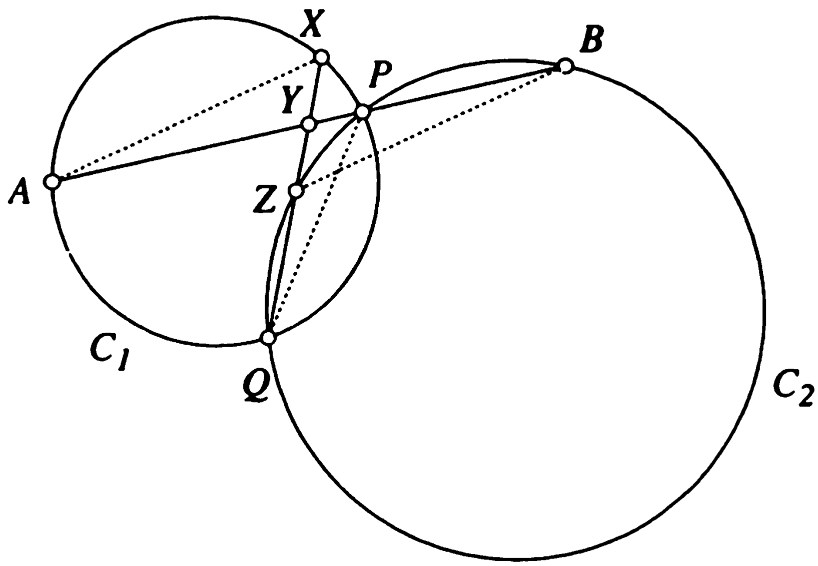

Figure 2

In circle C_2, we have $\angle ZBP = \angle ZQP$; and in circle C_1, we have $\angle PQX = \angle PAX$. Thus, we obtain $\angle ZBA = \angle BAX$. (So BZ is parallel to AX.) The triangles AXY and BZY are then congruent, because by hypothesis $AY = YB$ and angles AYX and YAX are respectively equal to BYZ and YBZ. This congruence gives us $XY = ZY$, which is what we want.

Aliter: We use a standard property of intersecting chords of a circle. If AB and CD are two chords of a circle intersecting at O either internally or externally, then $AO \cdot OB = CO \cdot OD$.

In the figure, AP and XQ are two chords of the circle C_1, intersecting externally at Y. So

$$AY \cdot YP = XY \cdot YQ \qquad (1)$$

Similarly, BP and QZ are two chords of the circle C_2 intersecting at externally at Y. So

$$YP \cdot YB = YZ \cdot YQ. \tag{2}$$

The left hand side expressions of (1) and (2) are equal because it is given that $AY = YB$. Therefore the right-hand expressions are equal. This give, on canceling the factor YQ, the desired relation $XY = YZ$.

53. Join OP and produce it to meet CD in L. We see that it suffices to prove that PL is perpendicular to CD, for in that case H would lie on the altitude PL implying that O, P, H are collinear.

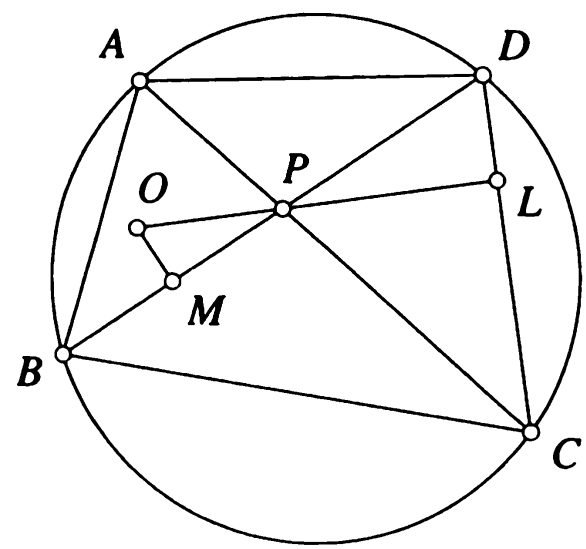

Figure 3

Since $ABCD$ is a cyclic quadrilateral, we have $\angle PDL = \angle PAB$. But O is the circumcentre of triangle APB. So $\angle PAB = \frac{1}{2}\angle POB$. If OM is the perpendicular bisector of BP, then

$$\frac{1}{2}\angle POB = \angle POM = \angle 90^\circ - \angle OPM = 90^\circ - \angle DPL.$$

Putting these results together, we get $\angle PDL = 90^0 - \angle DPL$. Thus $\angle PLD = 180^\circ - (\angle PDL + \angle DPL) = 90^\circ$; i.e., PL is perpendicular to CD.

54. We shall show that the locus of all such points is the union of the circumcircle and the orthocentre of the triangle ABC.

Let P be any point in the cone determined by two sides, say, BA and BC. Using the sine rule in the triangles PAC and PBC, we get

$$\angle CAP = \alpha \text{ or } 180^\circ - \alpha.$$

Similarly, using the triangles CAP and BAP, we also get

$$\angle ACP = \beta \quad \text{or} \quad 180^\circ - \beta.$$

Consider the case $\angle CAP = \alpha$ and $\angle ACP = 180^\circ - \beta$.

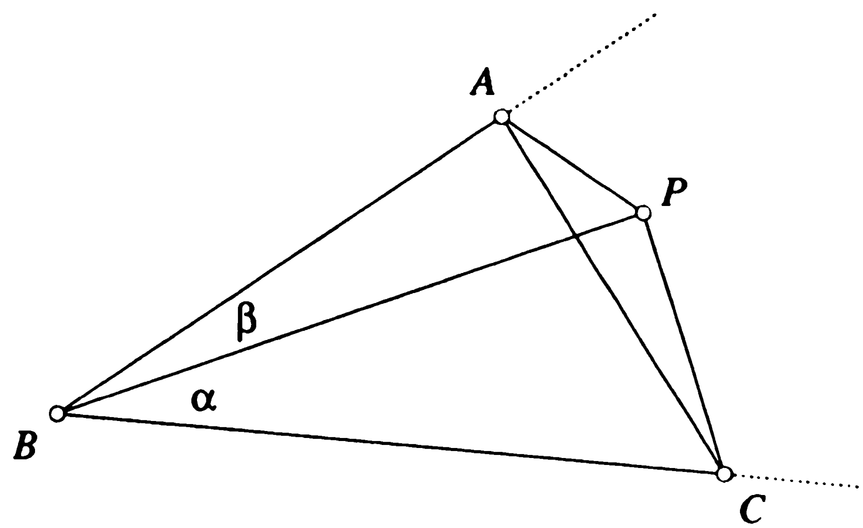

Figure 4a

Here we get,

$$\angle APC = 180^\circ - (\alpha + 180^\circ - \beta) = \beta - \alpha.$$

Again the triangles BPC and BPA give

$$\angle BAP = \angle BCP \quad \text{or} \quad \angle BAP = 180^\circ - \angle BCP.$$

If $\angle BAP = \angle BCP = \gamma$, then the sum of the angles of the quadrilateral is equal to $2\beta + 2\gamma$. This implies that $\beta + \gamma = 180^\circ$. Since β and

γ are angles of a triangle, this is impossible. If $\angle BAP = 180° - \angle BCP = 180° - \gamma$, then we get $-2\beta + 360° = 180°$. Hence $\beta = 90°$. This forces that $\angle PCA = 90°$ and AP is a diameter of the circle through A, B, C and P, i.e., P is on the circumcircle of ABC. Similarly, we can dispose off the case $\angle CAP = 180° - \alpha$, $\angle ACP = \beta$. Finally consider the case, $\angle CAP = 180° - \alpha$ and $\angle ACP = 180° - \beta$. Considering the triangle ACP, we see that

$$\angle APC = 180° - \angle ABC.$$

Similarly, the case $\angle CAP = \alpha$, $\angle ACP = \beta$ gives that $\angle APC$ and $\angle ABC$ are supplementary angles. Thus, A, B, C and P are concyclic.

On the other hand, suppose P is in the cone determined by the lines, say, CB and AB extended. Since

$$\angle PBC + \angle PAC = \angle PBA + \angle PCA = 180°,$$

it follows that $\angle ABC$ and $\angle APC$ are supplementary angles. Thus, triangles ABC and APC, and hence triangles ABC and BPC, have the same circumradii. Now sine rule gives

$$\angle CPB = \beta \text{ or } 180° - \beta, \angle APB = \gamma \text{ or } 180° - \gamma.$$

Also, if $\angle BAP = \alpha$, then $\angle BCP = \alpha$ or $180° - \alpha$. Consider the case

$$\angle CPB = \beta, \angle APB = 180° - \gamma \text{ and } \angle BCP = \alpha.$$

Then

$$\angle APC = \beta + 180 - \gamma, \ \angle PAC + \angle PCA = \beta + \gamma + 2\alpha$$

and hence $\beta + \gamma + 2\alpha = \gamma - \beta$ or $\alpha + \beta = 0$ which is impossible. If $\angle BCP = 180° - \alpha$, then we have

$$\angle APC = \beta + 180 - \gamma, \ \angle PAC + \angle PCA = \beta + \gamma + 180.$$

Then we would have,

$$\gamma - \beta = \beta + \gamma + 180$$

which is impossible. Similarly we can dispose off the cases

$$\angle CPB = 180^\circ - \beta,\ \angle APB = \gamma,\ \angle BCP = \alpha \text{ or } 180^\circ - \alpha.$$

Finally if

$$\angle CPB = \beta,\ \angle APB = \gamma,\ \angle BCP = 180^\circ - \alpha,$$

then again we get

$$\angle APC = \beta + \gamma,\ \angle PAC + \angle PCA = 180^\circ + \beta + \gamma.$$

This forces $2(\beta + \gamma) = 0$ which is impossible. We conclude that the only possibility is

$$\angle APB = \gamma, \angle CPB = \beta \text{ and } \angle BCP = \alpha.$$

In this case, we get

$$\angle APC = \beta + \gamma, \angle PAC + \angle PCA = 2\alpha + \beta + \gamma.$$

This gives us

$$\alpha = 90^\circ - (\beta + \gamma).$$

Thus $\beta + \alpha = 90^\circ - \gamma$ and $\alpha + \gamma = 90^\circ - \beta$. These imply that AP is perpendicular to CB and CP is perpendicular to AB. Hence P is the orthocentre.
Similarly we can consider other regions determined by BA and CA or BC and AC.
Finally if P is a point inside the triangle, we can show that P is the orthocentre of the triangle ABC in the similar way.
Thus if P is any point satisfying the hypothesis, then either P is the orthocentre of the triangle ABC or P must be on the circumcircle of the triangle ABC.

Aliter:

We need to know the following facts about three equal circles intersecting in a common point. If three congruent (that is, equal) circles C_1, C_2, C_3 have a common point P and A, B, C are the other three points of intersections, then

(a) the circumcircle of triangle ABC has the same radius as the three circles;

and

(b) the point P is the orthocentre of triangle ABC.

A brief proof of (a) and (b) follows:

Let X, Y, Z be the centres of the circles C_1, C_2, C_3 respectively. Complete the quadrilaterals $PXBZ$ and $PXCY$, join AP and ZY. Observe that $PXBZ$ and $PXCY$ are rhombuses and so ZB is parallel and equal to YC. Hence so are BC and ZY. Since AP is perpendicular to ZY, AP is perpendicular to BC. Similarly BP and CP are perpendicular to CA and AB respectively. Hence P is the orthocentre of triangle ABC. This proves (b).

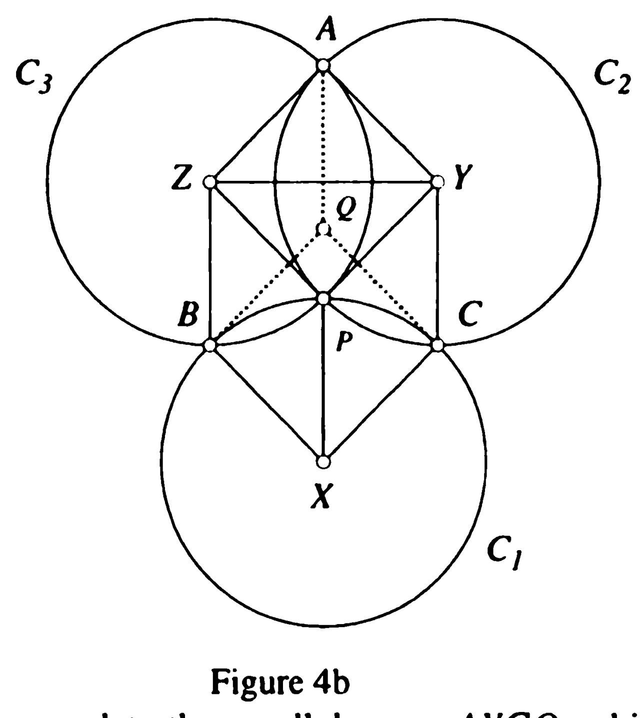

Figure 4b

To prove (a), complete the parallelogram $AYCQ$, which is in fact a rhombus. So $AQ = CQ$. It is easily seen that $AZBQ$ is also a rhombus. So $AQ = BQ$. Thus Q is circumcentre of triangle ABC and its radius $(= AQ = CY)$ is the same as that of each of the three circles. Note that we can have a configuration of three equal circles such that P falls outside triangle ABC, but statements (a) and (b) are still true.

Coming to the problem, let (XYZ) denote the circle through any three non collinear points X, Y, Z. It is given that three equal circles pass through P. Hence by (a) above, the four circles (PAB), (PBC), (PCA) and (ABC) are congruent to one another. Observe that either the three circles (PAB), (PBC), (PCA) coincide [and hence coincide with (ABC)] or the three circles are all distinct passing through the point P. Thus either P is on the circumcircle of ABC or P is the orthocentre of ABC.

55. Let K, L, M be the feet of perpendiculars from A to CD, BD and BC respectively. (Note that one foot is outside the circle in general.)

We have $AL = x, AM = y, AK = z$. Let $\beta = \angle ADB = \angle ACB$, $\gamma = \angle ABC = \angle ADK$, $\delta = \angle ABD = \angle ACD$.

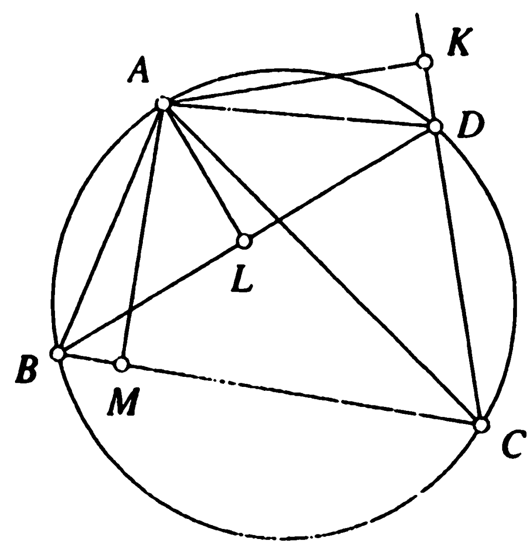

Figure 5

Now

$$\frac{BC}{y} + \frac{CD}{z} = \frac{BM + MC}{y} + \frac{CK - DK}{z}$$

$$\begin{aligned}
&= \frac{BM}{y} + \frac{MC}{y} + \frac{CK}{z} - \frac{DK}{z} \\
&= \cot\gamma + \cot\beta + \cot\delta - \cot\gamma \\
&= \cot\beta + \cot\delta \\
&= \frac{DL}{x} + \frac{BL}{x} \quad \text{(from triangles } ADL \text{ and } ABL) \\
&= \frac{BD}{x}.
\end{aligned}$$

Thus we have the desired relation.

56. We denote areas of triangles ABC, quadrilaterals $ABCD$, etc. by $[ABC]$, $[ABCD]$ etc. Join PQ and draw one of the diagonals, say BD. We use the fact that the median of a triangle bisects its area (Why?).

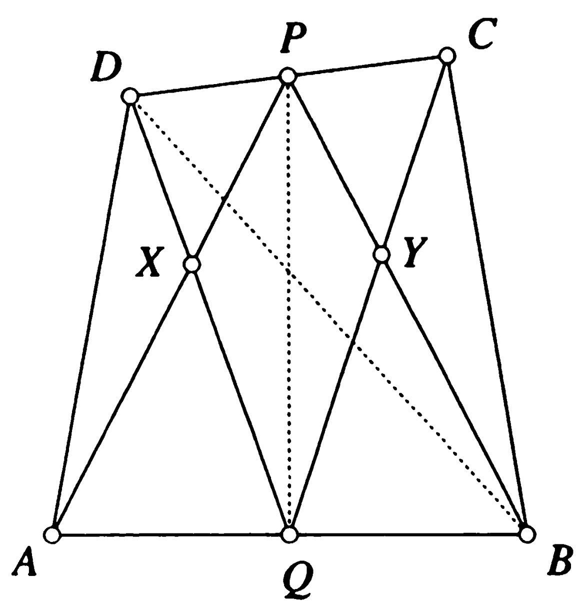

Figure 6a

From triangles DAB (with median DQ) and BCD (with median BP), we have

$$[ADQ] = [BDQ] \text{ and } [BPC] = [BPD].$$

Adding, we have

$$\begin{aligned}[ADQ] + [BPC] &= [BDQ] + [BPD] \\ &= [BPDQ] = [BPQ] + [DPQ] \\ &= [APQ] + [CPQ],\end{aligned}$$

since PQ is a median of both the triangles APB and CQD. Writing in terms of smaller areas, we have

$$[AXQ] + [AXD] + [BYC] + [PYC]$$

$$= [AXQ] + [PXQ] + [CPY] + [QPY].$$

On cancellation, this yields, $[ADX] + [BCY] = [PXQY]$.

If $ABCD$ is a concave quadrilateral and the points P, Q, X, Y are located as in the problem (see figure 6b), then by a similar argument, we arrive at the relation $|[ADX] - [BCY]| = [PXQY]$, where the left hand side denotes the modulus of the difference of areas. The proof is left to the reader.

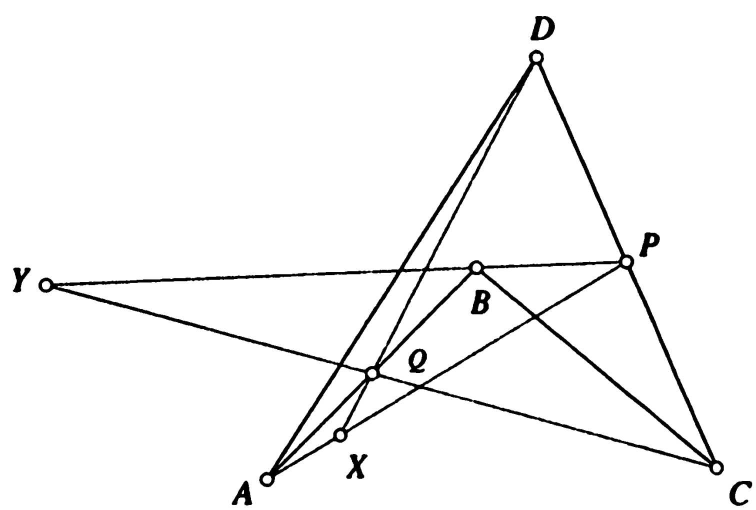

Figure 6b

17. We use the fact that the areas of two triangles having the same height are in the ratio of their bases. We also use some simple properties of equal fractions.

Specifically, if $a/b = c/d$, then each fraction is also equal to $(a+c)/(b+d)$ as well as $(a-c)/(b-d)$.

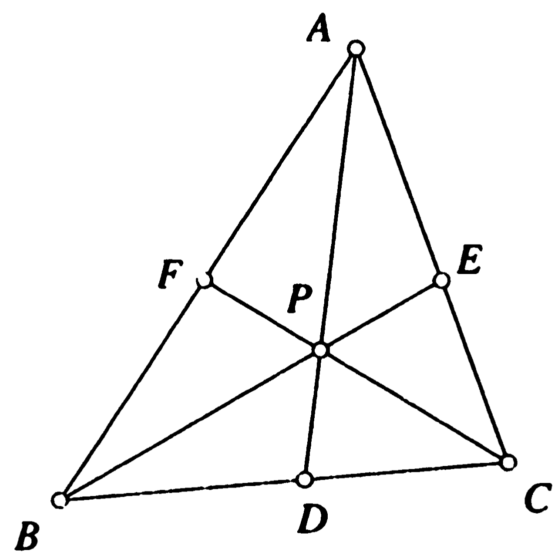

Figure 7

Now

$$\frac{[ACF]}{[BCF]} = \frac{AF}{FB} = \frac{[APF]}{[BPF]}.$$

So

$$\frac{AF}{FB} = \frac{[ACF]-[APF]}{[BCF]-[BPF]} = \frac{[ACP]}{[BCP]}. \quad (1)$$

Similarly from

$$\frac{[ABE]}{[CBE]} = \frac{AE}{EC} = \frac{[APE]}{[CPE]}$$

one obtains

$$\frac{AE}{EC} = \frac{[ABE]-[APE]}{[CBE]-[CPE]} = \frac{[ABP]}{[CBP]}. \quad (2)$$

From (1) and (2), by addition, we get

$$\frac{AF}{FB} + \frac{AE}{EC} = \frac{[ACP]+[ABP]}{[BCP]}. \quad (3)$$

Again,

$$\frac{AP}{PD} = \frac{[ABP]}{[DBP]} = \frac{[CAP]}{[DCP]} = \frac{[ABP] + [ACP]}{[BCP]} \tag{4}$$

From (3) and (4), we have the desired result.

Aliter: (Vineet Kahlon) Applying Ceva's theorem to the Cevians AD, BE, CF which are concurrent at P, we have

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Therefore,

$$\frac{AE}{EC} = \frac{AF}{FB} \cdot \frac{BD}{DC}.$$

Hence

$$\frac{AF}{FB} + \frac{AE}{AC} = \frac{AF}{FB}\left(1 + \frac{BD}{DC}\right) = \frac{AF}{FB} \cdot \frac{BC}{DC}. \tag{1}$$

Now applying Menelaus' Theorem to triangle ABD, whose sides are cut by the line FPC, we have

$$\frac{AF}{FB} \cdot \frac{BC}{DC} \cdot \frac{DP}{PA} = +1.$$

Consequently,

$$\frac{AF}{FB} \cdot \frac{BC}{DC} = \frac{AP}{PD}. \tag{2}$$

Comparing (1) and (2), we have the desired relation.

58. Let A, B be the centres of the circles with radii a and b respectively and touching externally at L. For the problem to make sense, obviously we have to take a direct common tangent(PQ) and not the common tangent at L (justify!). Let C be the centre of the circle with radius c touching these two circles externally at M and N and a direct common tangent PQ at R.

First we consider the two circles with centres A and C and evaluate PR (in terms of a and c). Note that PR is also the projection of AC on the common tangent.

If we draw CK parallel to RP to meet AP in K, then clearly $CKPR$ is a rectangle and

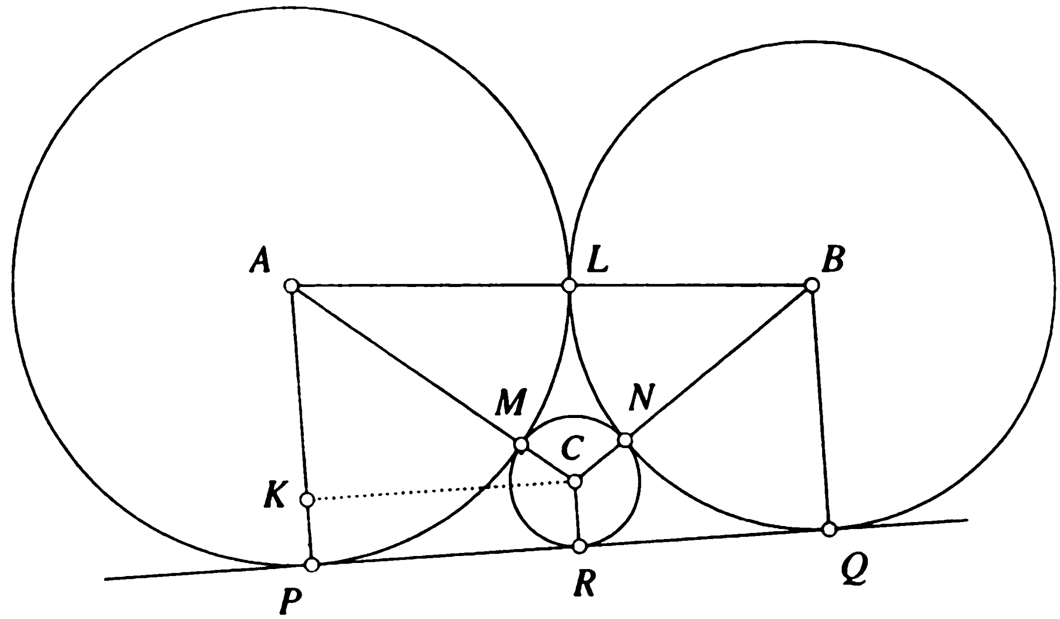

Figure 8

$$\begin{aligned} PR^2 &= CK^2 = AC^2 - AK^2 \\ &= (AM + MC)^2 - (AP - KP)^2 \\ &= (a + c)^2 - (AP - CR)^2 \\ &= (a + c)^2 - (a - c)^2 = 4ac. \end{aligned}$$

Therefore $PR = 2\sqrt{ac}$.

But then, we can apply the same argument to the other two pairs of circles namely those with centres B and C and those with centres A and B and obtain the relations

$$RQ = 2\sqrt{bc} \quad \text{and} \quad PQ = 2\sqrt{ab}.$$

But $PQ = PR + RQ$. Thus

$$2\sqrt{ab} = 2\sqrt{ac} + 2\sqrt{bc}.$$

Dividing through by $2\sqrt{abc}$, we get

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{a}},$$

which is what we wanted.

59. In the figure below h_a, h_b, m_a are given by AK, BL, AM respectively. Draw MT parallel to BL meeting AC in T. Then as MT is parallel to BL in triangle BLC, we get $MT = \frac{1}{2}BL = \frac{1}{2}h_b$. This enables us to construct triangle AKM and AMT. We can then complete the triangle ABC.

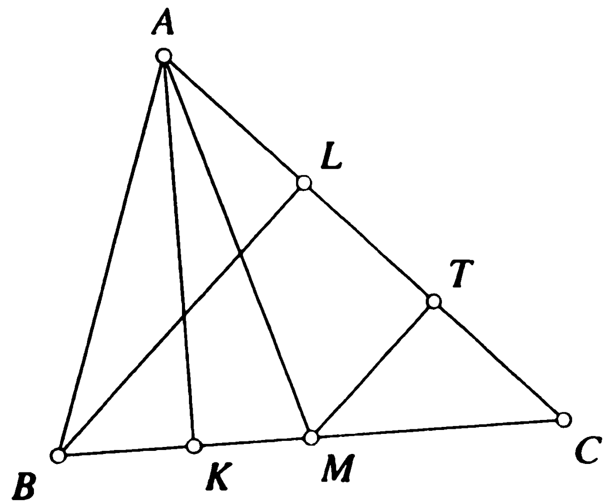

Figure 9

The construction is as follows: First construct triangle AKM from the measurements $AK = h_a, AM = m_a$ and $AKM = 90°$. With AM as a diameter draw a semicircle outwardly and with M as centre draw an arc of radius $\frac{1}{2}h_b$ to cut the semicircle at T. Join AT and produce AT to meet KM produced at C. Choose a point B on MK such that $BM = MC$. Join AB. Then ABC is the required triangle.

60. First we analyze the problem. Suppose we have located points B and C such that triangle ABC has the given perimeter, say $2s$. If the ex-circle opposite A touches the rays $\vec{AB}$ and $\vec{AC}$ at X and Y, we know that $AX = AY = s$. This gives us a method of construction as follows: First locate points X and Y on $\vec{AP}$ and $\vec{AQ}$ respectively such that $AX = AY = s$. Draw the perpendiculars to AP and AQ at X and Y respectively to meet at O. With O as centre and OX (or OY) as radius draw a circle. Draw a tangent from L to the circle to meet the rays $\vec{AP}$ and $\vec{AQ}$ in B and C respectively. (There are two tangents from L to the circle; choose the one nearer to A.) Then ABC is the required triangle.

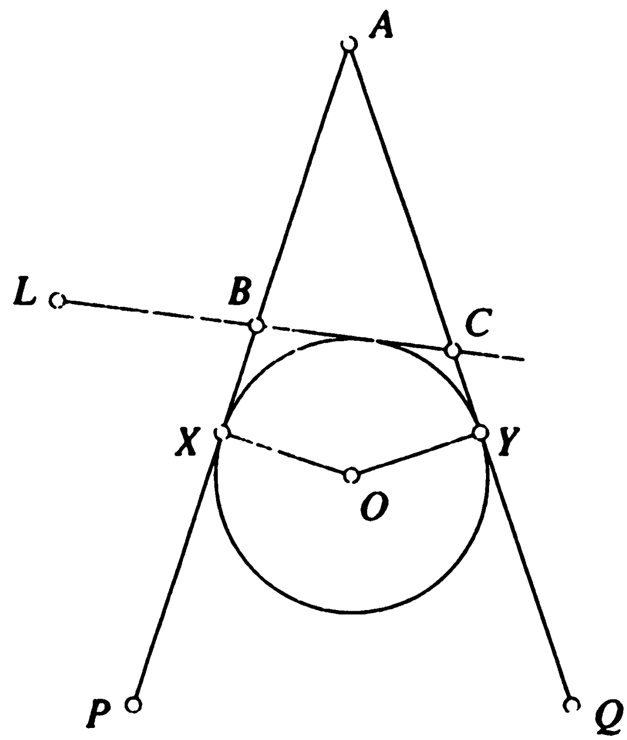

Figure 10

61. Join ID and AI. We first show that triangles BDG and BIA are similar. We have

$$\angle DBG = \frac{B}{2} = \angle IBA. \quad (1)$$

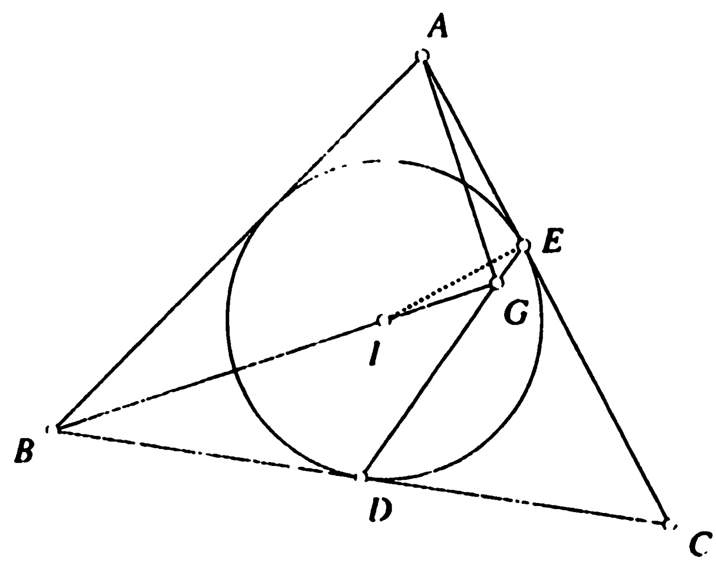

Figure 11

Further $\triangle CDE$ is isosceles, because $CD = CE$. So

$$\angle CDE = \angle CED = 90^\circ - \frac{C}{2}.$$

Therefore

$$\angle BDG = 180^\circ - \angle CDE = 90^\circ + \frac{C}{2}.$$

Also

$$\begin{aligned}\angle BIA &= 180^\circ - \angle ABI - \angle BAI \\ &= 180^\circ - \frac{B}{2} - \frac{A}{2} = 90^\circ + \frac{C}{2}.\end{aligned}$$

Thus,

$$\angle BDG = \angle BIA. \qquad (2)$$

From (1) and (2), it follows that, $\triangle BDG$ is similar to $\triangle BIA$. Therefore

$$\frac{BD}{BI} = \frac{BG}{BA}.$$

This fact along with the relation $\angle DBI = \angle GBA (= B/2)$ implies that triangles DBI and GBA are similar. Consequently, $\angle BDI = \angle BGA$. But $\angle BDI = 90^\circ$. So $\angle BGA = 90^\circ$. That is, BG is perpendicular to AG.

Aliter:

Join IE. We show that $IGEA$ is a cyclic quadrilateral. As in the previous method,

$$\angle CED = \angle CDE = 90^\circ - \frac{C}{2}.$$

Therefore

$$\angle AEG = 180^\circ - \angle DEC = 180^\circ - \left(90^\circ - \frac{C}{2}\right) = 90^\circ + \frac{C}{2}.$$

Also

$$\angle AIG = \angle ABI + \angle BAI = \frac{B}{2} + \frac{A}{2}.$$

So,

$$\angle AEG + \angle AIG = 90^\circ + \frac{C}{2} + \frac{B}{2} + \frac{A}{2} = 180^\circ.$$

This implies that $IGEA$ is a cyclic quadrilateral. So

$$\angle AGI = \angle AEI = 90^\circ.$$

That is AG is perpendicular to GI, as required.

62. Since $PXAY$ is a parallelogram, we have PX is parallel to AY. As AY is perpendicular to the tangent AB, it follows that PX is also perpendicular to AB. Since AB is a chord of the circle with centre X we conclude that PX is in fact the perpendicular bisector of AB

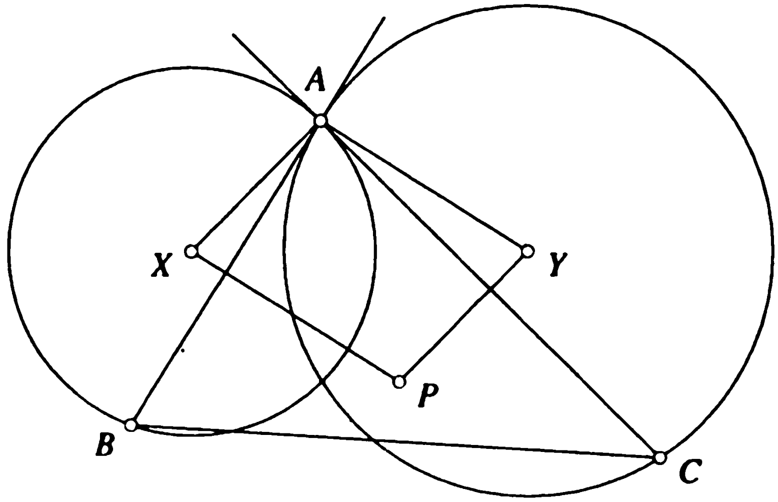

Figure 12

Similarly, PY is the perpendicular bisector of AC. Thus the perpendicular bisectors of two sides, AB and AC, meet in P, consequently P is the circumcentre of triangle ABC.

63. From the relation $BI^2 = BX \cdot BA$ we see that BI is a tangent to the circle passing through A, X, I at I. Hence

$$\angle BIX = \angle BAI = \frac{A}{2}. \qquad (1)$$

[Alternatively, one observes tat in triangles BIX and BAI, $\angle IBX$ is common and $BI/BX = BA/BI$. Consequently the two triangles are similar, implying (1).]

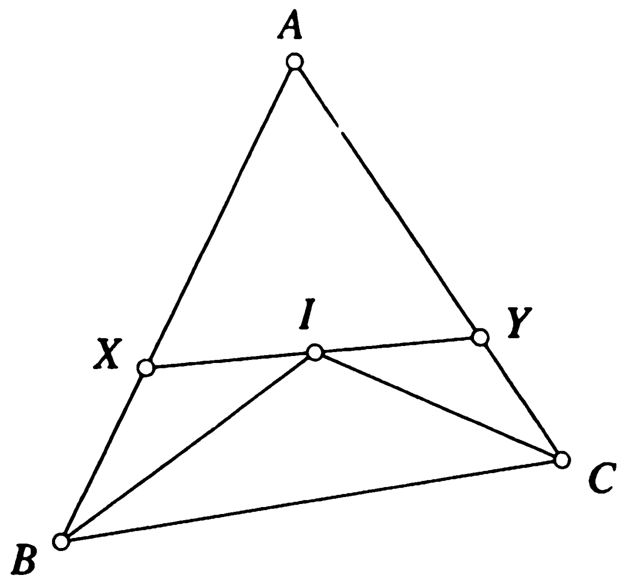

Figure 13

Similarly, from the relation $CI^2 = CY \cdot CA$ we obtain

$$\angle CIY = \angle CAI = \frac{A}{2}. \tag{2}$$

It is known that

$$\angle BIC = 90^\circ + \frac{A}{2}. \tag{3}$$

From (1), (2), (3) and the fact that X, I, Y are collinear, we obtain

$$\frac{A}{2} + \frac{A}{2} + \left(90^\circ + \frac{A}{2}\right) = 180^\circ.$$

Solving we get $A = 60^\circ$.

64. Let AI meet BC in K. Join IS. We do some angle-chasing now. Since AK is parallel to ST, we have

$$\angle STB = \angle AKB = \angle KCA + \angle KAC = C + \frac{A}{2}.$$

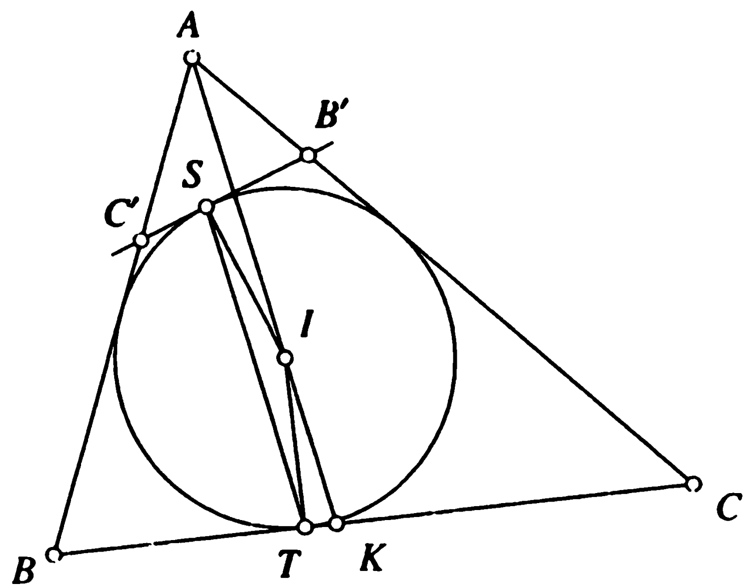

Figure 14

So

$$\angle STI = 90^\circ - \angle STB = 90^\circ - \left(C + \frac{A}{2}\right).$$

But $\angle TSI = \angle STI$ since SIT is an isosceles triangle. Therefore

$$\angle C'ST = 90^\circ - \angle TSI = C + \frac{A}{2}.$$

In the quadrilateral $BTSC'$.

$$\begin{aligned}\angle SC'B &= 360^\circ - (\angle C'BT + \angle BTS + \angle TSC') \\ &= 360^\circ - \left(B + C + \frac{A}{2} + C + \frac{A}{2}\right) \\ &= 360^\circ - (A + B + C + C) = 180^\circ - C.\end{aligned}$$

Hence

$$\angle AC'B' = 180^\circ - \angle SC'B$$

$$= 180^\circ - (180^\circ - C) = C.$$

Similarly, $\angle AB'C' = B$. Thus it follows that triangles ABC and $AB'C'$ are similar.

65. From the given relation, we have

$$A_1A_2 \cdot A_1A_3 + A_1A_2 \cdot A_1A_4 = A_1A_3 \cdot A_1A_4. \qquad (1)$$

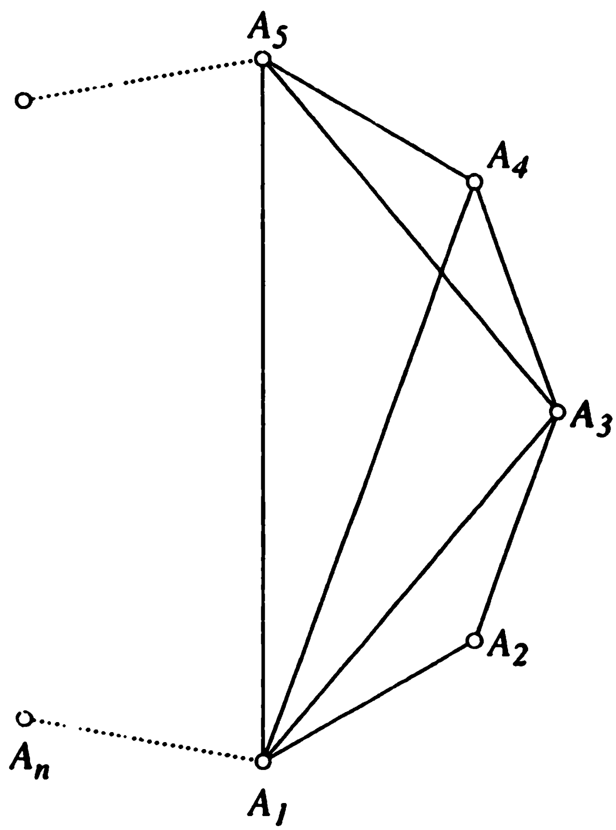

Figure 15

Also in the cyclic quadrilateral $A_1A_3A_4A_5$, we have, by Ptolemy's theorem,

$$A_4A_5 \cdot A_1A_3 + A_3A_4 \cdot A_1A_5 = A_3A_5 \cdot A_1A_4. \qquad (2)$$

Since $A_1A_2...A_n$ is a regular polygon, we have

$$A_1A_2 = A_4A_5,\ A_1A_2 = A_3A_4,\ A_1A_3 = A_3A_5.$$

Comparing (1) and (2), we have

$$A_1A_4 = A_1A_5.$$

Since the two diagonals A_1A_4 and A_1A_5 are equal, it follows that there must be the same number of vertices between A_1 and A_4 as between A_1 and A_5. That is the polygon must be 7-sided, that is $n = 7$.

Aliter:
If O is the centre of the circle in which $A_1A_2.....A_n$ is inscribed and θ is the angle which each side of the polygon subtends at O then using the relation

$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$$

obtain an equation in θ. Solve the equation to get $\theta = \frac{2\pi}{7}$. This means $n = 7$.

66. Let K, L, M be the points of contact of the semicircle with the sides BC, CD, DA respectively. Join OK, OL, OM, OC and OD where O is the centre of the circle as well as the midpoint of AB.

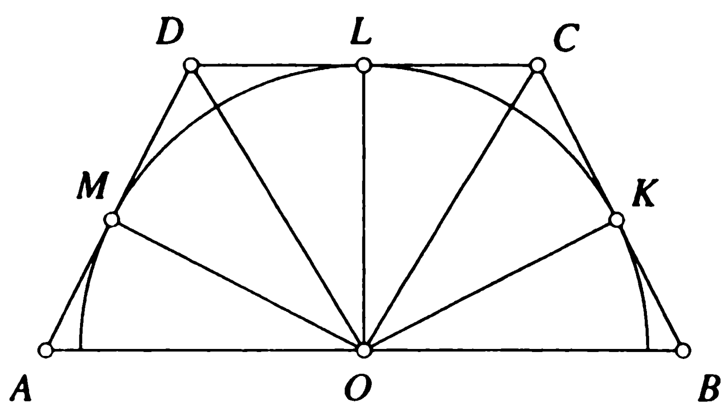

Figure 16

In the right angled triangles AOM and BOK, we have $AO = BO$ and $OM = OK$. Hence these two triangles are congruent. Thus

$$\angle AOM = \angle BOK = \alpha, \ \angle DOM = \angle DOL = \beta \text{ and}$$

$$\angle COL = \angle COK = \gamma.$$

Adding up the angles formed at O, we obtain $2\alpha + 2\beta + 2\gamma = 180°$ and so $\alpha + \beta + \gamma = 90°$.

In triangles AOD and BCO, we have

$$\angle OAD = 90° - \angle AOM = 90° - \alpha;$$

and

$$\angle CBO = 90° - \angle KOB = 90° - \alpha.$$

Similarly

$$\angle AOD = \alpha + \beta = 90° - \gamma;$$

and

$$\angle BCO = 90° - \angle COK = 90° - \gamma.$$

Therefore, triangle AOD is similar to triangle BCO.

Consequently

$$\frac{AO}{AD} = \frac{BC}{BO}.$$

So

$$AD \cdot BC = AO \cdot BO = \frac{1}{4}AB^2.$$

That is

$$AB^2 = 4AD \cdot BC.$$

67. Assume that $DE = DF$. First we find a suitable expression for DF using the cyclic quadrilateral $BDPF$. Observe that BP is a diameter of the circumscribing circle. By the sine rule (applied to triangle DBF), we have

$$\frac{DF}{\sin \angle FBD} = 2(\text{radius of the circumcircle of triangle DBF})$$

That is,

$$\frac{DF}{\sin B} = BP,$$

or

$$DF = BP \sin B$$

Similarly $DE = CP \sin C$.

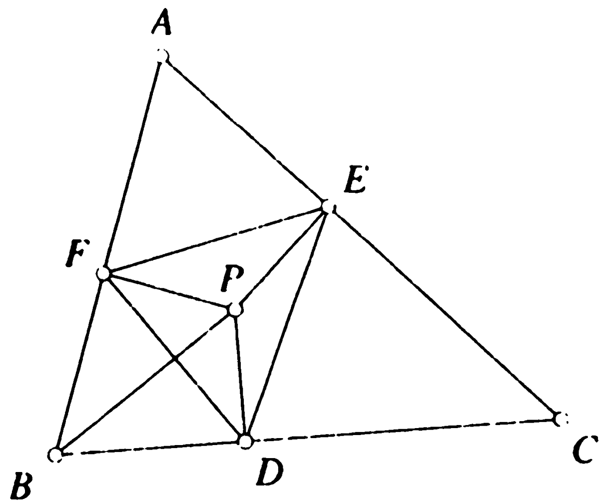

Figure 17

Since $DE = DF$, we have

$$\frac{BP}{PC} = \frac{\sin C}{\sin B} = \frac{c}{b}$$

a constant. Thus for $DE = DF$, we should have that $BP : PC$ is a constant ratio, $c : b$. But we know that the locus of such a point P is a circle. Thus the locus of P(for triangle DEF to be isosceles) is the union of three arcs of circles that lie in the interior of the triangle. These three circles must have a common point(because $DE = DF$ and $FE = FD$ implies $EF = ED$) and this is the only point for which triangle DEF is equilateral.

Note

[The circle corresponding to $DE = DF$ can be obtained as follows: Divide BC internally and externally in the ratio $c : b$ say at K and L respectively. Draw a circle with KL as a diameter. This is then the required circle, called the Appolonius circle.]

68. Let K, L, M be the centres of the three circles of equal radii, meeting in a common point O, and pairwise touching some side of the triangle ABC in which the lie. Since the three circles have equal radii, we see that O is equidistant from the points K, L, M and so is

the circumcentre of the triangle KLM. Also for the same reason, the sides of the triangle KLM are parallel to the corresponding sides of the triangle ABC. (For instance, KL is parallel to AB, as K and L are equidistant from AB.)

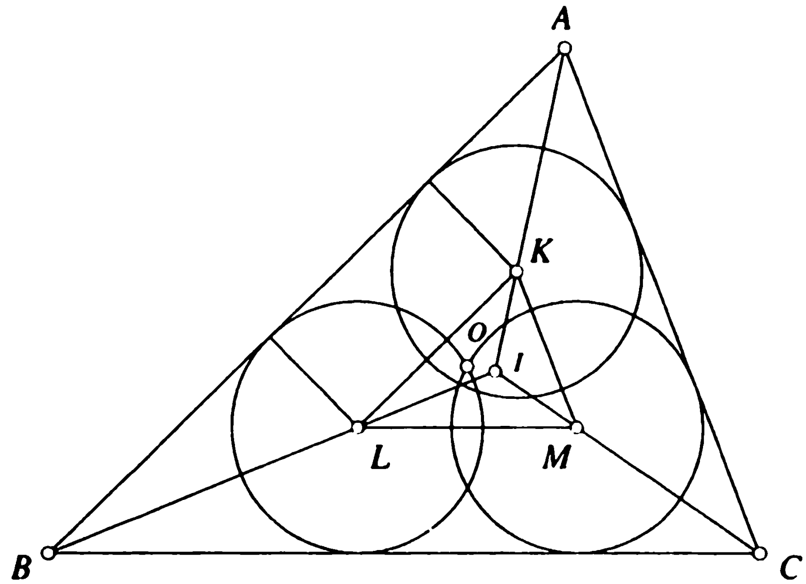

Figure 18

Further AK, BL, CM are the bisectors of angles A, B, C of triangle ABC. So not only they meet in I, the incentre of the triangle ABC but also KI, LI, MI are the bisectors of the angles of triangle KLM, implying that I is also the incentre of triangle KLM.

It follows, from the relation

$$\frac{IK}{IA} = \frac{IL}{IB} = \frac{IM}{IC},$$

that triangle KLM is homothetic to triangle ABC with respect to I, the centre of homothety. (This simply means that triangle ABC is a dilation (or an enlargement) of triangle KLM as seen from I).

A property of homothety is that the centre of homothety and any two corresponding points of the homothetic figures are collinear. Here, in particular, I and the circuncentres of triangles KLM and ABC have to be collinear. This is precisely what was to be proved.

69. Let S_1 touch the circle S at K, the rays AB and AC at M and L respectively. We have $PL = PM = PK = r_1$ (as P is the centre of S_1) and $R = OK = OP + r_1$, where R is the circumradius of triangle ABC (Note that O, the midpoint of the hypotenuse BC is the circumcentre of triangle ABC.) From the figure, it is clear that $AMPL$ is a square with side r_1.

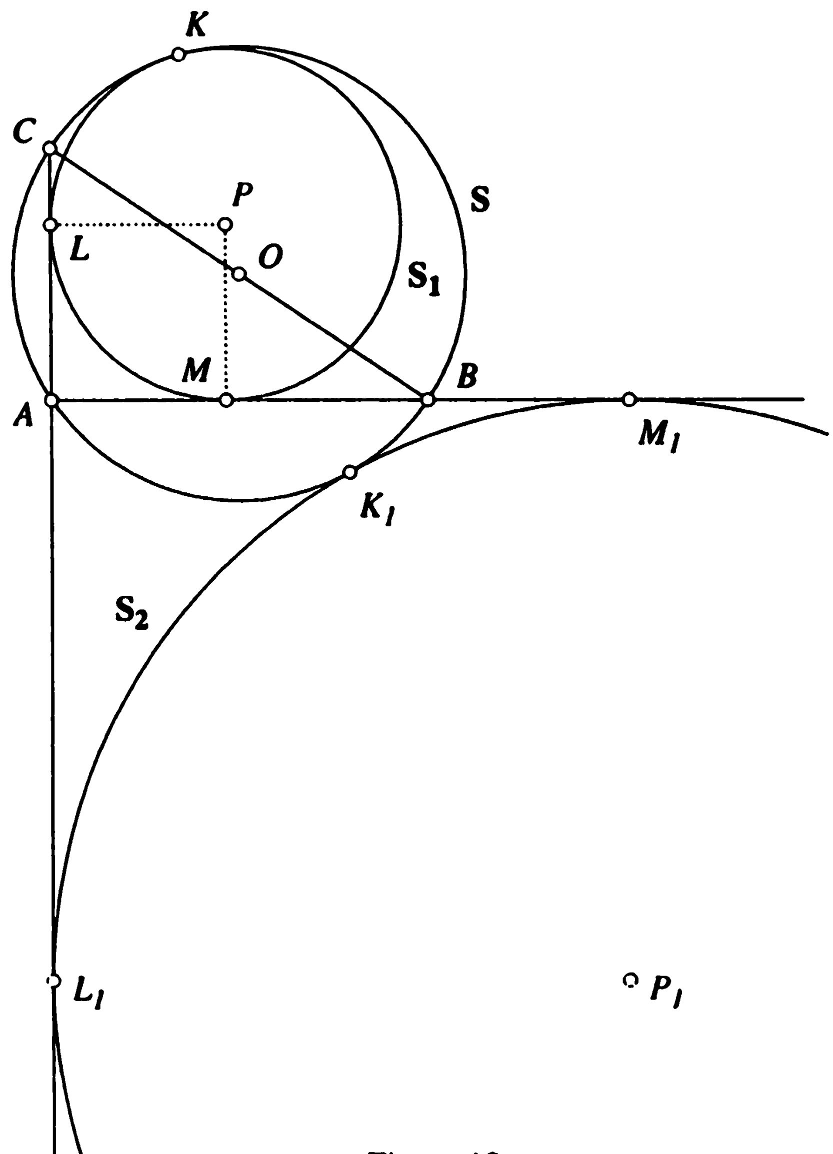

Figure 19

So

$$BM = AB - AM = c - r_1; \quad \text{and} \quad LC = AC - AL = b - r_1.$$

Therefore from triangles BMP and CLP, we have

$$PB^2 = PM^2 + MB^2 = r_1^2 + (c - r_1)^2$$

and

$$PC^2 = PL^2 + LC^2 = r_1^2 + (b - r_1)^2.$$

Applying Appolonius' theorem to triangle PBC, in which PO is a median, we get

$$PB^2 + PC^2 = 2(PO^2 + CO^2).$$

That is,

$$r_1^2 + (c - r_1)^2 + r_1^2 + (b - r_1)^2 = 2[(R - r_1)^2 + R^2].$$

Using the fact that $R = a/2$ and $a^2 = b^2 + c^2$, if we solve the above equation for r_1, we obtain $r_1 = b + c - a$.

Similarly, working with S_2 we obtain $r_2 = b + c + a$.

Hence

$$\begin{aligned} r_1 r_2 &= (b + c - a)(b + c + a) \\ &= (b + c)^2 - a^2 = b^2 + c^2 + 2bc - a^2 \\ &= 2bc = 4\left(\frac{1}{2}bc\right) \\ &= 4[ABC]. \end{aligned}$$

Aliter

Choose A as the origin, AB and AC as the x-axis and y-axis respectively. Let $B = (b, 0)$ and $C = (0, c)$. Then the circumcentre of triangle ABC which is at the midpoint of BC is given by $O = \left(\frac{b}{2}, \frac{c}{2}\right)$.

Any circle Γ which touches the positive x-axis and positive y-axis will have its centre at (r, r), where r is the radius of the circle. Now the equation to the circumcircle S of triangle ABC is

$$\left(x - \frac{b}{2}\right)^2 + \left(x - \frac{c}{2}\right)^2 = \left(\frac{a}{2}\right)^2.$$

The equation to Γ is $(x-r)^2+(y-r)^2 = r^2$. If the two circles S and Γ touch each other either internally (giving $\Gamma = S_1$) or externally (giving $\Gamma = S_2$), then we have

$$\left(r \pm \frac{a}{2}\right)^2 = \left(r - \frac{b}{2}\right)^2 + \left(r - \frac{c}{2}\right)^2,$$

giving $r = b + c \pm a$. Here $b + c - a$ is the radius of the circle S_1, namely, r_1 and $b + c + a$ is that of S_2, namely r_2.

Hence $r_1 r_2 = (b + c - a)(b + c + a) = 4$ (area ABC), as before.

70. Let O be the centre of the circle and P, Q the feet of perpendiculars from O to AC and BD. Clearly $OPEQ$ is a rectangle.

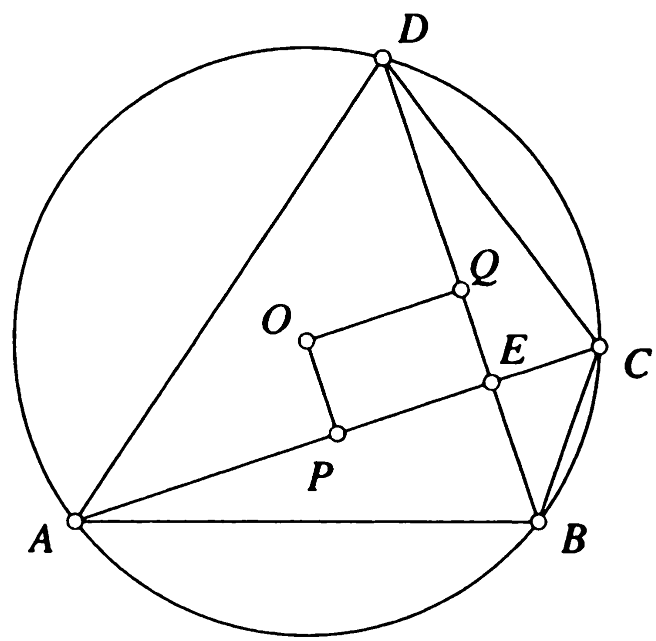

Figure 20

Now

$$\begin{aligned} EA^2 + EC^2 &= (EP + PA)^2 + (PC - PE)^2 \\ &= EP^2 + PA^2 + 2EP \cdot PA + \\ &\quad PC^2 + PE^2 - 2PC \cdot PE \\ &= 2(PA^2 + PE^2), \quad \text{because} \quad PA = PC. \end{aligned}$$

Similarly $EB^2 + ED^2 = 2(QD^2 + QE^2)$.

Therefore

$$\begin{aligned} EA^2 + EB^2 + EC^2 + ED^2 &= 2(PA^2 + PE^2) + 2(QD^2 + QE^2) \\ &= 2(PA^2 + OQ^2) + 2(QD^2 + OP^2) \\ &= 2(PA^2 + OP^2) + 2(QD^2 + OQ^2) \\ &= 2(OA^2 + OD^2) = 4R^2. \end{aligned}$$

71. We have (see figure) $PQ \cdot QR > BQ \cdot QC$, $QR \cdot RS > CR \cdot RD$, etc.

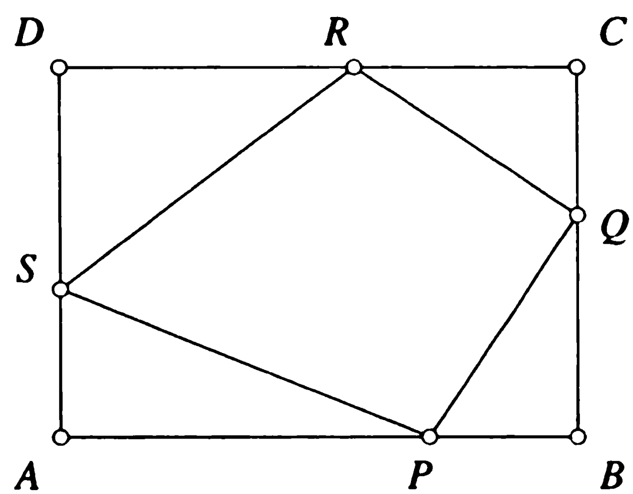

Figure 21

Therfore,

$$\begin{aligned} (PQ + QR + RS + SP)^2 &= PQ^2 + \cdots + \\ &\quad 2PQ \cdot QR + \cdots \\ &> (PB^2 + BQ^2) + \cdots + \\ &\quad +2BQ \cdot QC + \cdots \\ &= (PA + PB)^2 + (BQ + QC)^2 \\ &\quad +(CR + RD)^2 + (DS + SA)^2 \\ &= AB^2 + BC^2 + CD^2 + DA^2 \\ &= AC^2 + BD^2 = 2AC^2. \end{aligned}$$

Hence $PQ + QR + RS + SP > \sqrt{2}\, AC$.

72. Draw CQ such that $\angle PCQ = 60^\circ$ and $CP = CQ$.

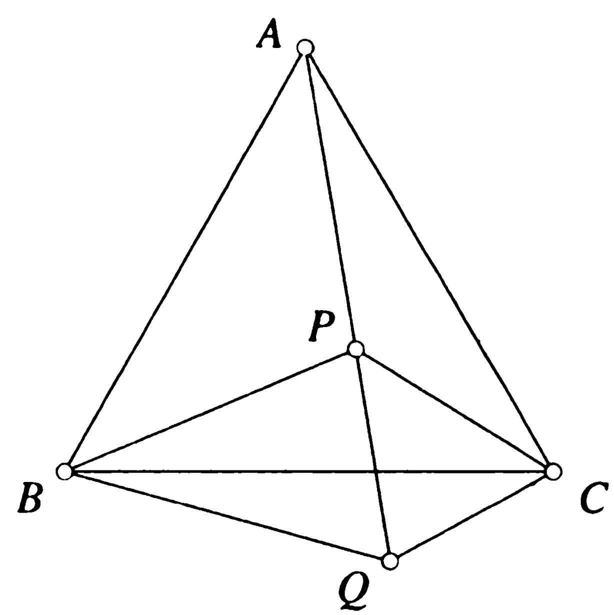

Figure 22

Then $\triangle PCQ$ is equilateral and therefore, $PQ = PC$. Also, in triangles APC and BQC:

$$AC = BC;\ PC = QC \quad \text{and} \quad \angle ACP = 60^\circ - \angle PCB = \angle BCQ.$$

The triangles are congruent. Therefore, $AP = BQ$. Substituting these in $AP^2 = BP^2 + CP^2$, we obtain

$$BQ^2 = BP^2 + PQ^2,$$

which implies $\angle BPQ = 90^\circ$ Therefore we obtain

$$\angle BPC = \angle BPQ + \angle QPC = 90^\circ + 60^\circ = 150^\circ.$$

73. Draw a line l parallel to BC through A and reflect AC in this line to get AD. Let CD intersect l in P. Join BD.

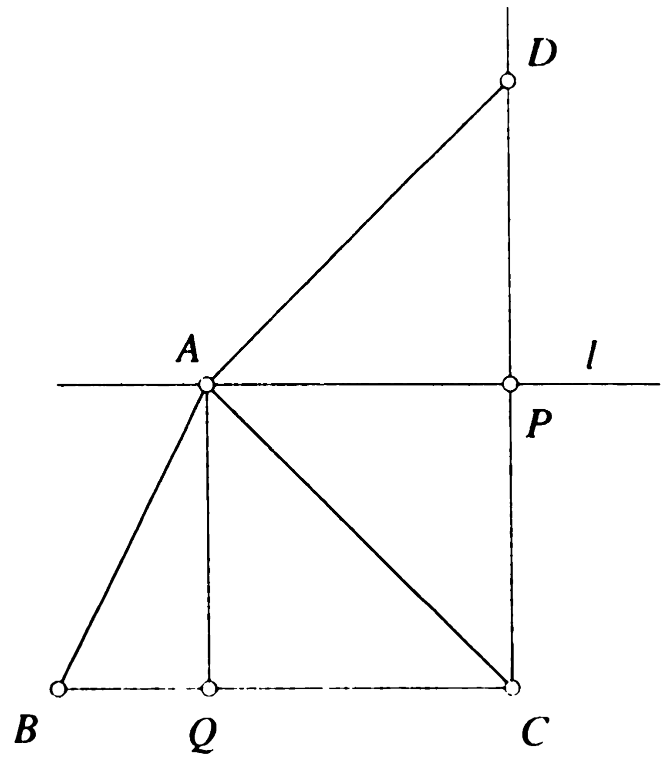

Figure 23

Observe that $CP = PD = AQ = h_a$, AQ being the altitude through A. We have

$$b + c = AC + AB = AD + AB \geq BD = \sqrt{CD^2 + CB^2}$$
$$= \sqrt{4h_a^2 + a^2},$$

which yields the result. Equality occurs if and only if B, A, D are collinear, i.e., if and only if $AD = AB$ (as AP is parallel to BC and bisects DC) and this is equivalent to $AC = BC$.

Alternatively, the given inequality is equivalent to

$$(b + c)^2 - a^2 \geq 4h_a^2 = \frac{16\Delta^2}{a^2},$$

where Δ is the area of the triangle ABC. Using the identity

$$16\Delta^2 = [\,(b + c)^2 - a^2\,][\,a^2 - (b - c)^2\,]$$

we see that the inequality to be proved is $a^2 - (b - c)^2 \le a^2$ (here we use $a < b + c$) which is true. Observe that equality holds if and only if $b = c$.

74. More generally, let $[BPF] = u, [BPC] = v$ and $[CPE = w$. Join AP. Let $[AFP] = x$ and $[AEP] = y$.

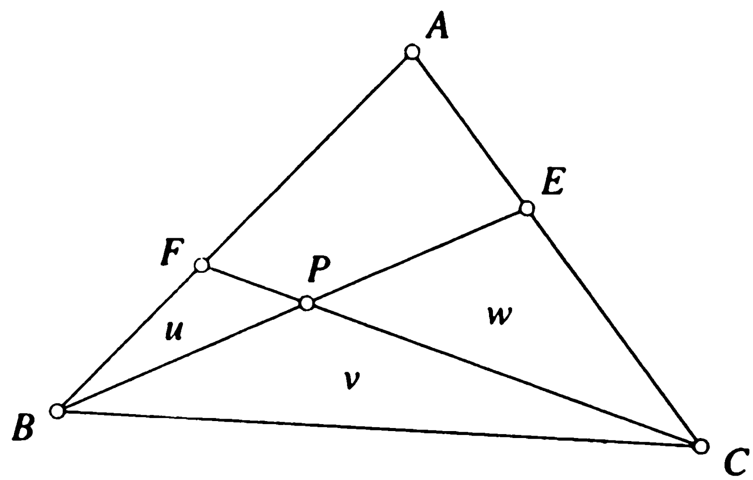

Figure 24

Using the triangles AFC and BFC, we get

$$\frac{x}{y+w} = \frac{FP}{PC} = \frac{u}{v}.$$

This gives the equation

$$vx - uy = uw.$$

Again using the triangles AEB and CEB we get another equation

$$wx - vy = -uw.$$

Solving these equations, we obtain

$$x = \frac{uw(u+v)}{v^2 - uw}, \quad y = \frac{uw(w+v)}{v^2 - uw}.$$

Hence we obtain

$$x + y = \frac{uw(u + 2v + w)}{v^2 - uw}.$$

Putting the values $u = 4, v = 8, w = 1$, we get $[AFPE] = 143$.

75. Let $AB = a, BC = b, CD = c, DA = d$. We are given that $abcd \geq 4$. Using Ptolemy's theorem and the fact that each diagonal cannot exceed the diameter of the circle we get $ac + bd = AC \cdot BD \leq 4$. But an application of AM-GM inequality gives

$$ac + bd \geq 2\sqrt{abcd} \geq 2\sqrt{4} = 4.$$

We conclude that $ac + bd = 4$. This forces $AC \cdot BD = 4$ giving $AC = BD = 2$. Each of AC and BD is thus a diameter. This implies that $ABCD$ is a rectangle. Note that

$$(ac - bd)^2 = (ac + bd)^2 - 4abcd \leq 16 - 16 = 0$$

and hence $ac = bd = 2$. Thus we get $a = c = \sqrt{ac} = \sqrt{2}$ and similarly $b = d = \sqrt{2}$. It now follows that $ABCD$ is a square.

3.4 Combinatorics

76. For $0 \leq j \leq 2$, let A_j denote the set of all integers between 1 and 300 which leave remainder j when divided by 3. Then $|A_j| = 100$ for $0 \leq j \leq 2$. If a, b, c is a 3-element subset of the given set $S = 1, 2, ..., 300$, then 3 divides $a + b + c$ if and only if

    (i) all a, b, c are in A_0 or in A_1 or in A_2,

    (ii) one of the a, b, c is in A_0, another in A_1, and the third one in A_2.

    The number of 3-element subsets of $A_j, 0 \leq j \leq 2$ is $\binom{100}{3}$. For each choice of a in A_0, b in A_1 and c in A_2, we get a 3-element subset such that 3 divides $a + b + c$. Thus the total number of 3-element subsets $\{a, b, c\}$ such 3 divides $a + b + c$ is equal to

$$3\binom{100}{3} + 100^3 = 1495100.$$

77. We count the 3-element subsets $\{a, b, c\}$ such that 4 does not divide abc. This is possible if and only if either all the three are odd numbers or any two of them are odd and the other is an even number not divisible by 4. There are 10 odd numbers in the set $\{1, 2, 3, \ldots, 20\}$ and 5 even numbers not divisible by 4. Thus the number of 3-element subsets $\{a, b, c\}$ such that 4 does not divide abc is equal to $\binom{10}{3} + 5\binom{10}{3} = 345$. The number of 3-element subsets is $\binom{20}{3} = 1140$. Thus the number of 3-element subsets such that the product of these elements is divisible by 4 is equal to $1140 - 345 = 795$.

78. Since each A_i contains 4 elements, totally we get 24 elements of which some may be repeated. But each element is repeated 4 times as each element belongs to exactly 4 of the $A_i's$. Hence there are $24/6 = 6$ distinct elements in S.

    Since $S = B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_n$, and each B_i consists of 2 elements, this union accounts for $2n$ elements. But each element appears exactly 3 times. Thus the number of distinct elements in S is also equal to $2n/3$.

Therefore $2n/3 = 6$ which gives $n = 9$.

79. This makes repeated use of the pigeon-hole principle. As there are 65 balls and two boxes, one of the boxes must contain at least 33 balls (as otherwise the total number balls would be $\leq 32 + 32 = 64$). Consider that box (i.e., the one containing ≥ 33 balls). We have more than 33 balls and four colours (white, black, red, yellow) and hence there must be at least 9 balls of the same colour in the box. There are at most 4 different sizes available for these 9 balls of the same colour. For if there were 5 (or more) different sizes, then the collection of five five balls, all of different sizes would not satisfy the given property. Thus among these 9 balls (of the same colour and in the same box) there must be at least 3 balls of the same size.

80. If both the urns have the same number of balls, then we can empty both the urns in one operation. Else, we remove the same number of balls from each of the urns so that one of the urns contains exactly one ball. (If m and n denote the number of balls in the urns, and say $m > n$, then take out $n - 1$ balls from each.) We now double the number of balls of the urn which contains only one ball and remove one ball from each of the urn. This process decreases the number of balls in the other urn by 1. Continuing this way we reach a stage when both the urns contain one ball each whence we can empty the urns removing one ball from each of the two urns.

81. Pair off the elements of the set $\{1, 11, 21, 31, \ldots 541, 551\}$ as follows: $\{\{1, 551\}, \{11, 541\}, \ldots \{271, 281\}\}$. There are 28 such pairs and they account for all the numbers in the original set. So if the subset A has more than 28 elements, then A should contain both the elements of some pair, but then there is a contradiction since each pair above has the property that the two elements in the pair add up to 552. Thus A cannot have more 28 elements.

82. For each n, $0 \leq n \leq 10$, let A_n denote the set of integers between

1 to 100 which leave remainder n after division by 11. Then A_1 consists of 10 elements and A_n for $n \neq 1$ consists of 9 elements each. If $\{a, b\}$ is any two-element subset of $\{1, 2, 3, \ldots\ldots 100\}$, the 11 divides $a + b$ if and only if either both a and b are in A_0 or else a is in A_k and b is in A_{11-k} for some k, $1 \leq k \leq 10$.

Consider any set B with 48 elements. If B contains two elements from the set A_0, then we are done. Similarly if B contains an element from A_k and another from A_{11-k}, $1 \leq k \leq 10$ then again, their sum is divisible by 11. Thus B can contain one element from A_0, 10 from A_1 and 9 from the sets A_k for some 4 values of $k(\neq 10)$, say k_1, k_2, k_3, k_4, no two of which add up to 11.

But these account only for 47 elements. Hence there must be an element which is either in A_{10} or in A_{11-k_j}, $1 \leq j \leq 4$. Thus we can always find an element a in A_k and b in A_{11-k}. Here a, b are in B and 11 divides $a + b$.

83. We shall look at the problem from a general viewpoint. For any positive integer n, let T_n denote the number of permutations of $(P_1, P_2, P_3, \ldots, P_n)$ of $1, 2, 3, \ldots, n$ such that for each k, $1 \leq k \leq n$, $(P_1, P_2, P_3, \ldots, P_n)$ is not a permutation of $1, 2, \ldots, k$. We shall obtain a formula for T_n which expresses T_n in terms of $T_1, T_2, \ldots, T_{n-1}$ $(n > 1)$. (Such a relation is called a *recurrence relation* for T_n.)

Consider any permutation $(P_1, P_2, P_3, \ldots, P_n)$ of $1, 2, \ldots, n$. There is always a least positive integer k such that $(P_1, P_2, P_3, \ldots P_k)$ is a permutation of $1, 2, \ldots, k$. In fact k may be any integer in the set $\{1, 2, \ldots, n\}$; and those permutations for which $k = n$ are exactly the ones we wish to count. The number of permutations of $(1, 2, \ldots, n)$ for all of which k is the least positive integer satisfying the above property is $T_k \cdot (n-k)!$, by our definition of T_n. The second factor corresponds to the permutations of $k+1, k+2, \ldots, n$ which fill up the remaining $(n-k)$ places. Since there are $n!$

permutations in all, we obtain

$$
\begin{aligned}
n! &= \sum_{k-1}^{n} T_k \cdot (n-k)! \\
&= T_1 \cdot (n-1)! + T_2 \cdot (n-2)! + \ldots + T_{n-1} \cdot 1! + T_n \cdot 0!
\end{aligned}
$$

Hence

$$T_n = n! - T_1 \cdot (n-1)! - T_2 \cdot (n-2)! \cdots - T_{n-1} \cdot 1!.$$

Clearly

$$
\begin{aligned}
T_1 &= 1. \\
T_2 &= 2! - T_1 \cdot 1! = 2 - 1 = 1 \\
T_3 &= 3! - T_1 \cdot 1! - T_2 \cdot 1! = 6 - 2 - 1 = 3 \\
T_4 &= 4! - T_1 \cdot 1! - T_2 \cdot 2! - T_3 \cdot 1! \\
&= 24 - 6 - 2 - 3 = 13. \\
T_5 &= 5! - T_1 \cdot 4! - T_2 \cdot 3! - T_3 \cdot 2! - T_4 \cdot 1! \\
&= 120 - 24 - 6 - 6 - 13 = 71 \\
T_6 &= 6! - T_1 \cdot 5! - T_2 \cdot 4! - T_3 \cdot 3! - T_4 \cdot 2! - T_5 \cdot 1! \\
&= 720 - 120 - 24 - 18 - 26 - 71 = 461.
\end{aligned}
$$

Thus the required number is 461.

84. Since none of the 17 integers has a prime factor exceeding 10, all of them have the form $2^a 3^b 5^c 7^d$, where a, b, c and d are non-negative integers. The product two such numbers, say $2^a 3^b 5^c 7^d$ and $2^{a'} 3^{b'} 5^{c'} 7^{d'}$, is $2^{a+a'} 3^{b+b'} 5^{c+c'} 7^{d+d'}$. Thus if $a + a', b + b', c + c'$ and $d + d'$ are all even then the product would be a square. For this to happen the 4-tuples (a, b, c, d) and (a', b', c', d') should have the parity (that is to say a and a' should both be odd or both even,b and b' should be both odd or both even etc.). Since each of the numbers a, b, c and d can either be odd or even, the total number of patterns of the 4-tuples (a, b, c, d) is $2^4 = 16$. As we have seventeen 4-tuples (a, b, c, d), each corresponding to the 17

given numbers, it follows, by the pigeon-hole principle, that at least two of these seventeen 4-tuples should have the same parity. The product of the numbers corresponding to these 4-tuples will then be a square.

85. **a.** Since A contains $n+1$ elements of the set $\{1, 2, 3, \ldots, 2n\}$ some two of the $n+1$ element must be consecutive (Why ?). But then any two consecutive integers are relatively prime and we have the desired conclusion.

b. We give a proof by making use of the pigeon-hole principle. Write each of the $n+1$ numbers in the form $2^p \cdot q$, where q is an odd number and p a nonnegative integer. What are the possible values of q? Since the numbers of A come from the set $\{1, 2, 3, \ldots 2n\}$, we see that q can be any one of the n odd numbers $1, 3, 5, 7, \ldots 2n-1$. As there are $n+1$ numbers in A, there are $n+1$ values of q. Hence by the afore-mentioned principle, for some two numbers $a = 2^{p_1} \cdot q_1$ and $b = 2^{p_2} \cdot q_2$, we must have $q_1 = q_2$. Since $a \neq b$, p_1 is either greater than p_2 or less than p_2. In the former case b divides a, while in the latter case a divides b.

Remark: Strangely, this problem can be solved by induction also. The reader should try this method.

86. Any real number x can be written as $\tan\alpha$ for some angle α between -90° and 90°. Put $x_i = \tan\alpha$, $1 \leq i \leq 7$, where x_i, $1 \leq i \leq 7$ are the given real numbers. Suppose α_i's are arranged such that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_7$. Then $\alpha_7 - \alpha_1 < 180^\circ$ and hence $\alpha_{i+1} - \alpha_i < 30^\circ$ for some index i. This gives

$$0 < \tan(\alpha_{i+1} - \alpha_i) < \frac{1}{\sqrt{3}}.$$

But

$$\begin{aligned}\tan(\alpha_{i+1} - \alpha_i) &= \frac{\tan\alpha_{i+1} - \tan\alpha_i}{1 + \tan\alpha_{i+1}\tan\alpha_i} \\ &= \frac{x_{i+1} - x_i}{1 + x_{i+1}x_i}.\end{aligned}$$

Thus we get

$$0 < \frac{x_{i+1} - x_i}{1 + x_{i+1}x_i} < \frac{1}{\sqrt{3}}.$$

87. Let A, B, C, D, E, F be the six cities, all pair wise connected by rail-way or road. We indicate the rail way by r and road by m. Consider the roads emanating from a fixed city, say A. There are five connections, namely AB, AC, AD, AE and AF. Since there are only two modes of transport, by the pigeon-hole principle three of them must belong to the same type. We may assume without loss of generality that AB, AC, AD are rail-ways (see the figure).

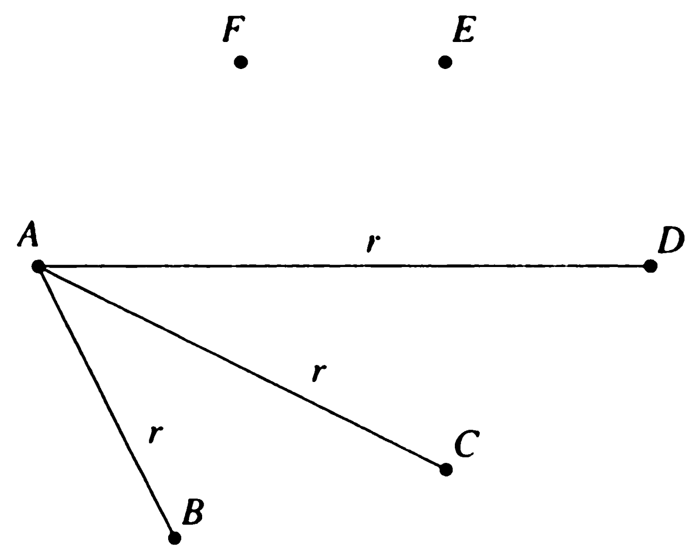

Figure 25

Now, if any two of the three cities B, C, D are connected by rail-way, then those two along with A constitute three cities mutually connected by the same mode of transport. Otherwise, all the three cities B, C, D would be connected pairwise by road in which case also we have the desired conclusion.

88. First we observe that each vertex can be present in at most 3 triangles, for, having chosen a vertex, there remains 7 points from which 3 pairs are possible. If there are 9 or more triangles, these account for at least 27 vertices, repetitions allowed. In that case,

one vertex has to occur in at least 4 triangles, a contradiction. Thus, there can be at most 8 triangles. The following example shows that 8 are possible; name the points as 1,2,3,4,5,6,7,8; the triangles are with vertices 123, 145, 167, 248, 368, 578, 256, 347.

89. Let us start counting 3-term *G.P.*'s with common ratios 2, 2^2, $2^3, \ldots$.

The 3-term *G.P.*'s with common ratio 2 are

$$1, 2, 2^2;\ 2, 2^2, 2^3;\ \cdots;\ 2^{n-2}, 2^{n-1}, 2^n.$$

They are $(n-1)$ in number. The 3-term *GP*'s with common ratio 2^2 are

$$1, 2^2, 2^4;\ 2, 2^3, 2^5;\ \cdots;\ 2^{n-4}, 2^{n-2}, 2^n.$$

They are $(n-3)$ in number. Similarly we see that the 3-term *GP*'s with common ratio 2^3 are $(n-5)$ in number and so on. Thus the number of 3-term *GP*'s which can be formed from the sequence $1, 2, 2^2, 2^3, \ldots, 2^n$ is equal to

$$S = (n-1) + (n-3) + (n-5) + \cdots$$

Here the last term is 2 or 1 according as n is odd or even. If n is odd, then

$$\begin{aligned} S &= (n-1) + (n-3) + (n-5) + \cdots + 2 \\ &= 2\left(1 + 2 + 3 + \cdots + \frac{n-1}{2}\right) \\ &= \frac{n^2-1}{4}. \end{aligned}$$

If n is even, then

$$S = (n-1) + (n-3) + \cdots + 1 = \frac{n^2}{4}.$$

Hence the required number is $(n^2-1)/4$ or $n^2/4$ according as n is odd or even.

90. First, note that A has 233 elements of which 116 are even and 117 are odd. B has 42 elements of which 21 are even and 21 are odd and $A \cap B$ has 14 elements. Therefore, required number is:

$$\begin{aligned} n &= |\{(a,b) : a \in A, b \in B, a+b \text{ is even }\}| \\ &\quad -|\{(a,b) : a \in A, b \in B, a = b\}| \\ &= |\{(a,b) : a \in A, b \in B, a \text{ is even, } b \text{ is even }\}| \\ &\quad +|\{(a,b) : a \in A, b \in B, a \text{ odd, } b \text{ odd }\}| \\ &\quad -|\{(a,b) : a \in A, b \in B, a = b\}| \\ &= 116 \times 21 + 117 \times 21 - 14 = 4879. \end{aligned}$$

91. If $100 \in A$ then we are done. So assume $A \subset \{1, 2, 3, \ldots, 99\}$. Consider the two-element subsets $\{1, 99\}$, $\{2, 98\}$, $\{3, 97\}$, $\ldots$, $\{49, 51\}$ along with the singleton set $\{50\}$. These fifty sets are disjoint, and their union is the set $\{1, 2, 3, \ldots, 99\}$ and the sum of the two numbers in each of the two-element set is 100. The hypothesis implies that A can contain at most one element from each of these fifty sets and since A has fifty elements it has to contain exactly one element from each of the fifty sets. Since $\{36, 64\}$ is one of the pairs and both 36 and 64 are squares we are done.

92. Suppose A has r elements, $0 \leq r \leq n$. Such an A can be chosen in $\binom{n}{r}$ ways. For each such A, the set B must necessarily have the remaining $(n - r)$ elements and possibly some elements of A. Thus, $B = (X \setminus A) \cup C$, where $C \subset A$. Hence B can be chosen in 2^r ways. Thus there are $\sum_{r=0}^{n} \binom{n}{r} 2^r = (1 + 2)^n = 3^n$ ways of choosing two sets A and B satisfying the given conditions. Among these choices, only in one case $A = B (= X)$, and in all other cases $A \neq B$. Since the order does not matter, we essentially have $(3^n - 1)/2$ pairs.

93. We start with the observation that $\left\lfloor \frac{801}{20} \right\rfloor = 40$ and hence some term must be at least 41. If we select nineteen numbers equal to 40 and one 41, then their sum is 801 but lcm is $40 \times 41 = 1640$.

If we take nineteen numbers equal to 41 and one 22, again they add up to 801 but *lcm* is $41 \times 22 = 902$. Any other combination containing 41 as a summand will have *lcm* equal to $41k$ for some $k > 1$; for observe that 41 is a prime and we cannot have a combination having only 41 as summands.

94. Since $x + 1$ divides $ax^2 + bx + c$, we must have $a + c = b$. Thus we have to count the number of triples (a, b, c) with the condition that a, b, c lie in the set $\{1, 2, 3, \ldots, 1999\}$, $a \neq c$ and $a + c = b$. If we take $a < c$, then for each a with $1 \leq a \leq 999$, c can take values from $a + 1$ to $1999 - a$. Thus for $a = 1$, c runs from 2 to 1998 giving 1997 ordered pairs (a, c) with $a < c$; for $a = 2$, c runs from 3 to 1997, giving 1995 pairs (a, c) with $a < c$, and so on. The number of ordered pairs (a, c) with $a < c$ and $a + c$ lying in the set $\{1, 2, 3, \ldots, 1999\}$ is thus equal to

$$1997 + 1995 + 1993 + \cdots + 1 = 999^2.$$

Similarly the number of pairs (a, c) with $c < a$ and $a + c$ lying in the set $\{1, 2, 3, \ldots, 1999\}$ is 999^2. Hence the required number of polynomials is $2 \cdot 999^2 = 1996002$.

95. Suppose that (a, b, c) is a subset of $\{1, 2, 3, \ldots, 63\}$ with $a+b+c < 95$. Then $(64-a, 64-b, 64-c)$ is a subset of $\{1, 2, 3, \ldots, 63\}$ with $(64 - a) + (64 - b) + (64 - c) = 192 - (a + b + c) > 192 - 95 = 97$. Conversely, if (a, b, c) is a subset of $\{1, 2, 3, \ldots, 63\}$ with $a + b + c > 97$, then $(64 - a, 64 - b, 64 - c)$ is such that $(64 - a) + (64 - b) + (64 - c) = 192 - (a + b + c) < 95$. Thus there is a one-one correspondence between 3-element subsets (a, b, c) with $a + b + c < 95$ and those such that $a + b + c > 97$. Hence the number of subsets with $a + b + c < 95$ is equal to that with $a + b + c > 97$. Thus the set of 3-element subsets (a, b, c) with $a + b + c > 95$ will contain those with $a + b + c > 97$ and a few more.

96. Let X be an n- element set and let B be a subset of X containing r elements. Thus there are $\binom{n}{r}$ choices for B. Hence there are 2^r

choices for A and $2^{n-r} - 1$ choices for C. Thus we obtain the total number of triples (A, B, C) such that $A \subset B \subset C$, but $B \neq C$ as $\sum_{r=0}^{n} 2^r \binom{n}{r}(2^{n-r} - 1)$ which simplifies to $4^n - 3^n$.

Aliter: Let us denote by 0 or 1 the absence or presence of an element of X in the sets A, B, C. For any fixed element of X, there are only four choices to conform with $A \subset B \subset C$, namely, 000, 001, 011, 111. Thus there are 4^n choices. But $B = C$ gives three choices, namely, 000, 011, 111. Hence there are 3^n triples (A, B, B). The number of triples (A, B, C) with $A \subset B \subset C$ but $B \neq C$ is therefore $4^n - 3^n$.

97. We fill up the 3×3 array at the left top (shown by dots in the adjacent figure) arbitrarily using the numbers $0, 1, 2, 3$. This can be done in 4^9 ways. The three numbers in the first row uniquely fix a. Similarly b, c, p, q, r are fixed uniquely (If a number n when divided by 4 leaves a remainder R, then $n + (4 - R)$ is divisible by 4 and $4 - R$ is in the set $\{0, 1, 2, 3\}$).

    It is also clear that $a + b + c$ and $p + q + r$ leave the same remainder modulo 4, since both are obtained by the same set of nine numbers adding row-wise and adding column-wise, modulo 4. Hence x is also fixed uniquely by the nine numbers originally chosen. Thus the number of arrays required is 4^9.

98. Delete any n rows containing *maximal* number of zeros. We claim that at most n zeros are left in the remaining n rows. For, if otherwise, there are at least $n + 1$ zeros left and so there are at least 2 zeros in some row, by the Pigeonhole Principle. Since we have deleted rows containing maximum number of zeros, each such row must contain at least 2 zeros. Hence we would have deleted at least $2n$ zeros. These along with $n + 1$ zeros would account for more than $3n$ zeros, a contradiction to the hypothesis. This proves our claim.

    Now remove the columns (numbering not more than n) containing the remaining zeros. By this process, we are removing all the $3n$ zeros in the desired manner.

99. Suppose $\{a, b, c, d\}$ is a group in which $a = (b + c + d)/3$. Then $a + b + c + d = 4a$. Hence, if such an n exists, then 4 divides $1 + 2 + \cdots + 4n$. However this sum is $2n(4n + 1)$. Thus a necessary condition for existence of such a set is that n be even.

We show that this condition is also sufficient; i.e., if $n = 2k$ for some k, then it is possible to partition $\{1, 2, 3, \dots, 8k\}$ into groups of 4 elements $\{a, b, c, d\}$ such that $a = (b + c + d)/3$. To this end, divide $\{1, 2, 3, \dots, 8k\}$ in to groups of 8 integers such that each group contains 8 consecutive integers. If $\{a + 1, a + 2, a + 3, \cdots, a + 8\}$ is one such set, we can divide this set into two sets of 4 integers each as follows:

$$\{a + 4, a + 1, a + 3, a + 8\}, \quad \{a + 5, a + 2, a + 6, a + 7\}.$$

The desired partition is obtained since

$$a + 4 = \frac{a + 1 + a + 3 + a + 8}{3}$$

and

$$a + 5 = \frac{a + 2 + a + 6 + a + 7}{3}.$$

100. (a) Let a, b, c be the sides of a triangle with $a + b + c = 1996$, and each being a positive integer. Then $a + 1, b + 1, c + 1$ are also sides of a triangle with perimeter 1999 because

$$a < b + c \quad \Longrightarrow \quad a + 1 < (b + 1) + (c + 1),$$

and so on. Moreover $(999, 999, 1)$ form the sides of a triangle with perimeter 1999, which is not obtainable in the form $(a + 1, b + 1, c + 1)$ where a, b, c are the integers and the sides of a triangle with $a + b + c = 1996$. We conclude that $f(1999) > f(1996)$.

(b) As in the case (a) we conclude that $f(2000) \geq f(1997)$. On the other hand, if x, y, z are the integer sides of a triangle with $x + y + z = 2000$, and say $x \geq y \geq z \geq 1$, then we cannot have $z = 1$; for otherwise we would get $x + y = 1999$ forcing x, y to have opposite parity so that $x - y \geq 1 = z$ violating triangle inequality for x, y, z. Hence $x \geq y \geq z > 1$. This implies that $x - 1 \geq y - 1 \geq z - 1 > 0$. We already have $x < y + z$. If $x \geq y + z - 1$, then we see that $y + z - 1 \leq x < y + z$, showing that $y + z - 1 = x$. Hence we obtain $2000 = x + y + z = 2x + 1$ which is impossible. We conclude that $x < y + z - 1$. This shows that $x - 1 < (y - 1) + (z - 1)$ and hence $x - 1, y - 1, z - 1$ are the sides of a triangle with perimeter 1997. This gives $f(2000) \leq f(1997)$. Thus we obtain the desired result.

3.5 Miscellaneous

101. Consider the smallest value among the 64 entries on the board. Since it is the average of the surrounding numbers, all those numbers must be equal to this number as it is the smallest. This gives some more squares with the smallest value. Continue in this way till all the squares are covered.

102. For every triplet (a, b, c) in T the triplet $(7 - c, 7 - b, 7 - a)$ is in T and these two are distinct as $7 \neq 2b$. Pairing off (a, b, c) with $(7-c, 7-b, 7-a)$ for each $(a, b, c) \in T$, 7 divides $abc + (7-a)(7-b)(7-c)$.

103. We begin with the observation that E and N can take the values only in the set $\{0, 5\}$. If $N = 5$, then there is a carry and the addition in 10's place cannot be true. Hence $N = 0, E = 5$. Consider the 1000's place. The addition there shows that there is a carry which must originate from 100's place. There can be at most a carry of 2 from 100's place to 1000's place. If the carry is 1, then O has to be 9 and $I = 0$. But since N is already 0, $I = 1$ and $O = 9$ with a carry 2 from 100's place. To account for the carry of 2, we must have $T \geq 5$. Since $E = 5, T \neq 5$. If $T = 6$, then R must be 7 or 8. But $R = 7$ gives $X = 0$ and $R = 8$ gives $X = 1$, which are impossible since $N = 0$ and thus $T \neq 6$.

If $T = 7$, then R must be from the set $\{5, 6, 8\}$. But R cannot be 5. Now $R = 6$ gives $X = 1$ and $R = 8$ gives $X = 3$. In the last case, F cannot be from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Hence $T = 8$ giving $R = 7, X = 4, F = 2$ and $S = 3$:

$$\begin{array}{r} 29786 \\ 850 \\ 850 \\ \hline 31486 \\ \hline \end{array}$$

104. Let S denote the set of all the 25 students in the class, X the set of swimmers in S, Y the set of all weight-lifters and Z the set of all cyclists. Since students in $X \cup Y \cup Z$ all get grades B and C and six students get grades D or E, the number of students in $X \cup Y \cup Z \leq 25 - 6 = 19$. Now assign one point to each of the 17 cyclists, 13 swimmers and 8 weight-lifters. Thus a total of 38 points would be assigned among the students in $X \cup Y \cup Z$. Note that no student can have more than 2 points as no one is all the three. Then we should have $|X \cup Y \cup Z| \geq 19$ as otherwise 38 points cannot be accounted for. (For example if there were only 18 students in $X \cup Y \cup Z$ the maximum number of points that could be assigned to them is 36). Therefore $|X \cup Y \cup Z| = 19$ and each student in $X \cup Y \cup Z$ is in exactly 2 of the sets X, Y, Z. Hence the number of students getting grade $A = 25 - 19 - 6 = 0$, i.e. no student gets A grade. Since there are 19 - 8 = 11 students who are not weight-lifters all these 11 students must be both swimmers and cyclists. (Similarly there are 2 who are both swimmers and weight-lifters and 6 who are both cyclists and weight-lifters).

105. Suppose E is wearing a white cap. Then D is lying and hence must be wearing a white cap. Since D and E both have white caps, A is lying and hence he must be wearing white cap. If C is speaking truth, then C must be having a black cap and B must be wearing a black cap as observed by C. But then B must observe a black cap on C. Hence B must be lying. This implies that B is wearing a white cap which is a contradiction to C's statement.

On the other hand if C is lying, then C must be wearing a white cap. Thus A, C, D and E are wearing white caps which makes B's statement true. But then B must be wearing a black cap and this makes C statement correct.

Thus E must be wearing a black cap. This implies that B is lying and hence must be having a white cap. But then D is lying and hence must be having a white cap since B and D have white caps. A is not saying the truth. Hence A must also be wearing a white cap. These together imply that C is truthful. Hence C must be wearing a black cap. Thus we have the following distribution:

A - white cap, B - white cap, C - black cap,
D - white cap, E - black cap.

106. The reader is expected to be familiar with the following simple properties of bijective functions:

a. If $f : A \longrightarrow A$ is a bijective function then there is a unique bijective function $g : A \longrightarrow A$ such that $fog = gof = 1_A$, the identity function on A. The function g is called the inverse of f and is denoted by f^{-1}. Thus,

$$fof^{-1} = I_A = f^{-1}of.$$

b. $foI_A = f = I_Aof$.

c. If f and g are bijections from A to A, then so are gof and fog.

d. If f, g, h are bijective functions from A to A and $fog = foh$, then $g = h$.

Apply f^{-1} at left to both sides to obtain $g = h$.

Coming to the problem, since A has n elements, we see that there are only finitely many (in fact, $n!$) bijective functions from A to A as each bijective function f gives a permutation of $\{1, 2, 3, \ldots, n\}$ by taking $\{f(1), f(2), \ldots, f(n)\}$. Since f is a bijective function from A to A, so is each of the functions in the sequence:

$$fof = f^2, fofof = f^3, \ldots, f^n, \ldots$$

All these cannot be distinct, Since there are only finitely many bijective functions from A to A. Hence for some two distinct positive integers m and n, $m > n$ say, we must have

$$f^m = f^n.$$

If $n = 1$, we take $M = m$, to obtain the result. If $n > 1$, multiply both sides by $(f^{-1})^{n-1}$, to get $f^{m-n+1} = f$. We take $M = m - n + 1$ to get the relation

$$f^M = f, (M > 1).$$

Note this means

$$f^M(i) = f(i) \quad \text{for all} \quad i \in A.$$

Aliter: Take any element r in the set A and consider the sequence of elements

$$r, f(r), (fof)(r), (fofof)(r), \ldots.$$

obtained by applying f successively. Since A has only n elements there must be repetitions in the above sequence. But when the first repetition occurs, this must be r itself; for, if the above sequence looks (for instance) like

$$r, a, b, c, d, e, c, \ldots.$$

where the first repetition is an element c other than r, this would imply

$$f(b) = c \quad \text{and} \quad f(e) = c,$$

contradicting the fact that f is a bijection. Thus for some positive integer $l_r \geq 1$, we have $f^{l_r}(r) = r$.

This is true for each r in the set $A = 1, 2, \ldots, n$. By taking M to be the *lcm* of $l_1, l_2, \ldots, l_r$ we get

$$f^M(r) = r \quad \text{for each} \quad r \in A.$$

[Note: If f itself is the identity function the above proof fails because each $l_r = 1$. But in this case we may take M to be any integer greater than or equal to 2].

107. Let $ABCDEF$ be an equiangular hexagon with side-lengths as 1, 2, 3, 4, 5, 6 in some order. We may assume without loss of generality that $AB = 1$. Let $BC = a$, $CD = b$, $DE = c$, $EF = d$, $FA = e$.

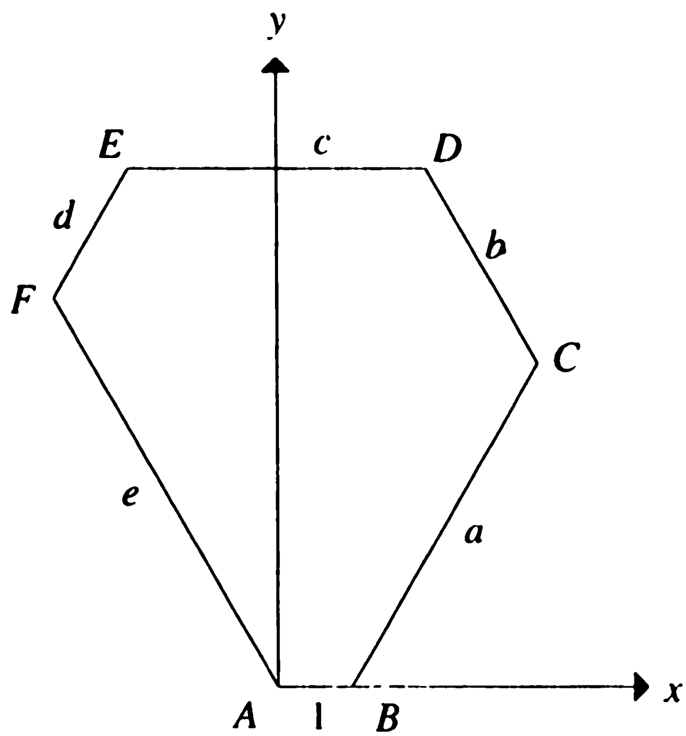

Figure 26a

Since the sum of all the angles of a hexagon is equal to $(6-2)\times 180^\circ = 720^\circ$ it follows that each (interior) angle must be equal to $720^\circ/6 = 120^\circ$. Let us take A as the origin, the positive x-axis along AB and the perpendicular at A to AB as the y-axis, as shown in the figure. We use the vector method. If the vector is denoted by (x, y), we then have

$$
\begin{aligned}
\vec{AB} &= (1,0), \vec{BC} = (a\cos 60^\circ, a\sin 60^\circ),\\
\vec{CD} &= (b\cos 120^\circ, b\sin 120^\circ),\\
\vec{DE} &= (c\cos 180^\circ, c\sin 180^\circ) = (-c, 0).\\
\vec{EF} &= (d\cos 240^\circ, d\sin 240^\circ),\\
\vec{FA} &= (e\cos 300^\circ, e\sin 300^\circ).
\end{aligned}
$$

This is because these vectors are inclined to the positive x-axis at angles $0, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$ respectively.

Since the sum of all these 6 vectors is $\vec{0}$, it follows that

$$1 + \frac{a}{2} - \frac{b}{2} - c - \frac{d}{2} + \frac{e}{2} = 0;$$

and

$$(a + b - d - e)\frac{\sqrt{3}}{2} = 0.$$

That is

$$a - b - 2c - d + e + 2 = 0, \qquad (1)$$

and

$$a + b - d - e = 0. \qquad (2)$$

Since $\{a, b, c, d, e\} = \{2, 3, 4, 5, 6\}$, in view of (2) we have

$$\begin{array}{ll} (i) & \{a, b\} = \{2, 5\}, \{d, e\} = \{3, 4\}, c = 6, \\ (ii) & \{a, b\} = \{3, 6\}, \{d, e\} = \{4, 5\}, c = 2, \\ (iii) & \{a, b\} = \{2, 6\}, \{d, e\} = \{3, 5\}, c = 4. \end{array}$$

[The possibility that $\{a, b\} = \{3, 4\}, \{c, d\} = \{2, 5\}$ in (i), for instance, need not be considered separately, because we can reflect the figure about $x = 1/2$ and interchange these two sets.]

Case(i):

Here $(a-b)-(d-e) = 2c-2 = 10$. Since $a = b = \pm 3, d-e = \pm 1$, this is not possible.

Case(ii):

Here

$$(a - b) - (d - e) = 2c - 2 = 2.$$

This is satisfied by

$$(a, b, d, e) = (6, 3, 5, 4).$$

Case (iii):

Here

$$(a - b) - (d - c) = 2c - 2 = 6.$$

This is satisfied by

$$(a, b, d, e) = (6, 2, 3, 5).$$

Hence we have (essentially) two different solutions:

$$(1, 6, 3, 2, 5, 4) \quad \text{and} \quad (1, 6, 2, 4, 3, 5).$$

it may be verified that (1) and (2) are both satisfied by these sets of values.

Aliter

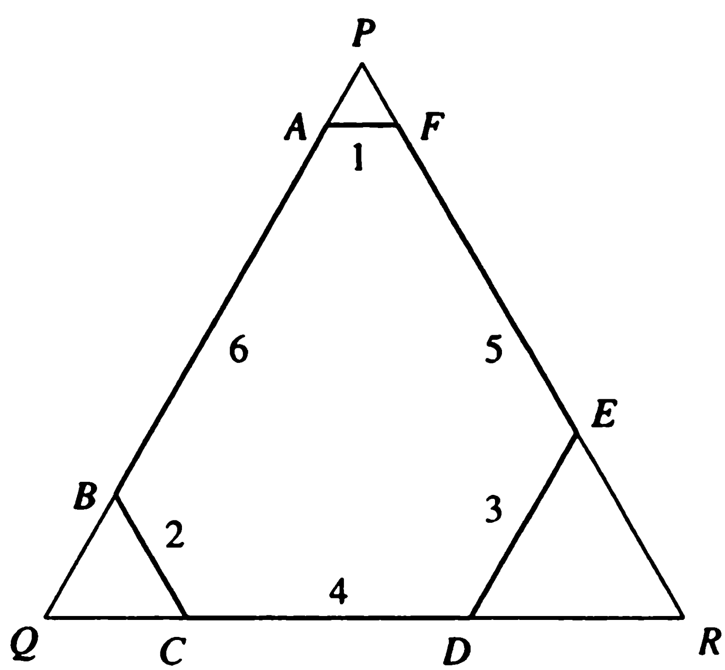

Figure 26b

Consider an equilateral triangle, of side 9 units. Remove from the three corners equilateral triangles of sides 1 unit, 2 units and 3 units respectively. The remaining portion is now an equiangular hexagon $ABCDEF$ with sides 1,6,2,4,3,5 as required.

108. Let $a_1, a_2, \ldots, a_{10}$ denote the weights of the 10 objects in decreasing order. It is given that $10 \geq a_1 \geq a_2 \geq \cdots a_{10} \geq 1$ and that

$a_1 + a_2 + \cdots + a_{10} = 20$. For each i, $1 \le i \le 9$, let $S_i = a_1 + \cdots + a_i$. (For example $S_1 = a_1, S_2 = a_1 + a_2$, etc.) Consider the 11 numbers $0, S_1, S_2, \ldots, S_9, a_1 - a_{10}$. Note that all these 11 numbers are non-negative and we have $0 \le a_1 - a_{10} < 10$ and $1 < S_i < 20$ for $1 \le i \le 9$. Now look at the remainders when these 11 numbers are divided by 10. We have 10 possible remainders and 11 numbers and hence by the pigeon-hole principle at least some two of these 11 numbers have the same remainder.

Case (i):

For some j, S_j has the remainder 0, i.e., S_j is multiple of 10. But since $1 < S_j < 20$ the only possibility is that $S_j = 10$. Thus we get a balancing by taking the two groups to be $a_1, \ldots, a_j$ and $a_{j+1}, \ldots, a_{10}$.

Case(ii):

Suppose $a_1 - a_{10}$ is a multiple of 10. But then since $0 \le a_1 - a_{10} < 10$ this forces $a_1 - a_{10} = 0$ which in turn implies that all the weights are equal and equal to 2 as they add up to 20. In this case taking any five weights in one group and the remaining in the other we again get a balancing.

Case(iii):

For some j and k, say $j < k$, we have that S_j and S_k have the same remainder, i.e., $S_k - S_j$ is a multiple of 10. But again since $0 < S_k - S_j < 20$ we should have $S_k - S_j = 10$, i.e., $a_k + a_{k-1} + \cdots + a_{j+1} = 10$ and we get a balancing.

Case(iv):

Suppose $(a_1 - a_{10})$ and S_j for some j $(1 \le j \le 9)$ have the same remainder, i.e., $S_j - (a_1 - a_{10})$ is a multiple of 10. As in the previous cases this implies that

$$S_j - (a_1 - a_{10}) = 10,$$

i.e.,

$$a_2 + a_3 + \cdots + a_j + a_{10} = 10.$$

Therefore $\{a_2, a_3, \ldots, a_j, a_{10}\}$ and $\{a_1, a_{j+1}, \ldots, a_9\}$ balance each other.

Thus in all cases the given 10 objects can be split into two groups that balance each other.

109. Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ and x_8 be the numbers written at the corners. Then, the final sum is given by

$$\sum_{i=1}^{8} x_i + x_1x_2x_3x_4 + x_5x_6x_7x_8 + x_1x_4x_5x_8$$
$$+x_2x_3x_6x_7 + x_1x_2x_5x_6 + x_3x_4x_7x_8.$$

Because there are fourteen terms in the above sum and each of the terms is $+1$ or -1, the sum will be zero only if some seven terms are $+1$ each and the remaining seven terms are -1 each.

But, the product of the fourteen terms is

$$(x_1x_2x_3x_4x_5x_6x_7x_8)^4 = (\pm 1)^4 = +1.$$

Therefore, it is impossible to have an odd number of -1's in the above sum.

We conclude that the desired arrangement is not possible.

110. One possible collection is $\{a, b, c\}$, $\{a, d, e\}$, $\{a, f, g\}$, $\{b, d, f\}$, $\{b, e, g\}$, $\{c, e, f\}$, $\{c, d, g\}$. Note that there could be other combinations obtained by permuting the letters. Without loss of generality a can be associated with three pairs b, c; d, e; f, g. Now b can be associated with d, f and e, g. The possible choices left for c are only the pairs e, f and d, g. This arrangement works.

Appendix A: Problems for practice

1. If $2x + 4y = 1$ prove that $x^2 + y^2 \geq (1/20)$.

2. If $x_1, x_2, ..., x_n$ are the roots of the equation

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_0 = 0,$$

   find the roots of the equation

$$a_0 x^n - a_1 x^{n-1} + a_2 x^{n-2} - ... + (-1)^n a_n = 0.$$

3. Suppose f is a function of two variables which satisfies

$$f(a, b) = f(a + b, b - a)$$

   for all real numbers a and b. If $g(x) = f(4^x, 0)$ prove that there exists a constant c such that $g(x + c) = g(x)$ for all real x.

4. If a, b, and c are positive real numbers such that $a + b + c = 1$, prove that $a^2 + b^2 + c^2 \geq \frac{1}{3}$.

5. Suppose $a_1, a_2, ...a_{n-1}$ are non-negative real numbers and consider the polynomial

$$p(x) = x^n + a_1 x^{n-1} + a_2 a^{n-2} + \cdots + a_{n-1} x + 1.$$

   Assume that the equation $p(x) = 0$ has n real roots. Prove that $p(2) \geq 3^n$.

6. If $x + (1/x) = -1$, find $x^{99} + (1/x^{99})$.

7. If a, b, c are nonzero numbers and are the roots of the equation

$$x^3 - ax^2 + bx - c = 0,$$

find a, b, c.

8. A person left home between 4 P.M. and 5 P.M., returned between 5 P.M.and 6 P.M. and found that the hands of his watch had exactly exchanged places. When did he go out?

9. Solve

$$\begin{aligned} \log_2 x + \log_4 y + \log_4 z &= 2; \\ \log_3 y + \log_9 z + \log_9 x &= 2; \\ \log_4 z + \log_1 6x + \log_1 6y &= 2. \end{aligned}$$

10. Find a finite sequence of 16 numbers such that:
a) it reads the same from left to right as from right to left;
b) the sum of any 7 consecutive terms is -1;
c) the sum of any 11 consecutive terms is $+1$.

11. Let n be an odd positive integer and $m_1, m_2, m_3, \ldots m_n$, a rearrangement of the numbers $1, 2, 3, \ldots, n$. Prove that the product $(m_1 - 1)(m_2 - 2)\ldots(m_n - n)$ is an even integer.

12. Triangle ABC is scalene with angle A having a measure greater than 90°. Determine the set of points D lie on the extended line BC, for which $|AD|^2 = |BD||CD|$ where $|XY|$ refers to the distance between the points X and Y.

13. Find the remainder when 19^{92} is divided by 92.

14. Given a triangle ABC, define x, y, z as follows:

$$\begin{aligned} x &= \tan((B - C)/2) \cdot \tan A/2 \\ y &= \tan((C - A)/2) \cdot \tan B/2 \\ z &= \tan((A - B)/2) \cdot \tan C/2. \end{aligned}$$

Prove that: $x + y + z + xyz = 0$.

15. Solve the following system of equations for real x, y, z,:

$$\begin{aligned} x + y - z &= 4 \\ x^2 - y^2 + z^2 &= -4 \\ xyz &= 6. \end{aligned}$$

16. Find the number of ways in which one can place the numbers $1,2,3....,n^2$ on the n^2 squares of an $n \times n$ chess board, one on each , such that the numbers in each row and column are in arithmetic progression.

17. If a, b, c, d are four non negative real numbers with $a+b+c+d = 1$, then show that $ab + bc + cd \leq \frac{1}{4}$.

18. N is a 50 digit number (in the decimal notation). All the digits except the 26th digit (from the left) are 1. If N is divisible by 13, find the 26th digit.

19. If a, b, c and d are four positive numbers show that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4.$$

20. Take any point P_1 on the side BC of a triangle ABC and draw the following chain of lines: P_1P_2 parallel to AC (P_2 on AB); P_2P_3 parallel to BC; P_3P_4 parallel to AB; P_4P_5 parallel to CA; and P_5P_6 parallel to BC. Show that P_6P_1 is parallel to AB.

21. Find all integers a such that the quadratic expression $(x + a)(x + 1991) + 1$ can be factored as a product $(x + b)(x + c)$ where b and c are integers.

22. Determine the set of integers n for which $n^2 + 19n + 92$ is a square.

23. In a quadrilateral $ABCD$, AB is parallel to CD, $AB \neq CD$, $AB = a$, $BC = b = AD$, $CD = c$ and $AC = d$. Show that $d^2 = b^2 + ac$.

24. If a, b, c, d are four positive real numbers, then show that

$$\begin{aligned}\frac{12}{a+b+c+d} &\leq \frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{a+d}+\frac{1}{b+c}+\\ &\quad +\frac{1}{b+d}+\frac{1}{c+d}\\ &\leq \frac{3}{4}\left[\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right].\end{aligned}$$

25. Given on acute-angled triangle ABC, let points A', B', C' be located as follows: A' is the point where altitude from A on BC meets the outwards facing semicircle drawn on BC as diameter. Points B', C' are located similarly. Prove that

$$[BCA']^2 + [CAB']^2 + [ABC']^2 = [ABC]^2,$$

where $[ABC]$ denotes the area of triangle ABC.

26. A square sheet of paper $ABCD$ is so folded that B falls on the mid-point M of CD. Prove that the crease will divide BC in the ratio 5:3.

27. Suppose P is any point inside a triangle ABC and s is the semi-perimeter of the triangle ABC, that is, $AB + BC + CA = 2s$. Prove that $s < AP + BP + CP < 2s$.

28. If the circumcentre and centroid of a triangle coincide, prove that the triangle must be equilateral. In general, show that if any two of the circumcentre, the incentre, the orthocentre and the centroid of a triangle coincide, then the triangle must be equilateral.

29. The cyclic octagon $ABCDEGFH$ has sides a, a, a, a, b, b, b, b respectively. Find the radius of the circle that circumscribes the octagon in terms of a and b.

30. Show that among all quadrilaterals of a given perimeter the square has the largest area.

31. A river flows between two houses A and B, the houses standing some distances away from the banks. Where should a bridge be built across the river so that a person going from A and B, using the bridge to cross the river may do so by the shortest path? Assume that the banks of the river are straight and parallel and the bridge is perpendicular to the banks.

32. Suppose P is an interior point of a triangle ABC and AP, BP, CP meet the opposite sides BC, CA, AB in D, E, F respectively. Find the set of all possible values the following quantities can take:

$$\frac{AP}{PD} + \frac{BP}{PE} + \frac{CP}{PE} \quad \text{and} \quad \frac{AP \cdot BP \cdot CP}{PD \cdot PE \cdot PF}.$$

33. Let ABC be an acute-angled triangle and CD, the altitude through C. If $AB = 8$ and $CD = 6$, find the distance between the midpoints of AD and BC.

34. Prove that the ten's digit of any power of 3 is even. [example: the ten's digit of $3^6 = 729$ is 2].

35. Suppose $A_1A_2...A_{20}$ is a 20-sided regular polygon. How many non-isosceles (scalene) triangles can be formed whose vertices are among the vertices of the polygon but whose sides are not the sides of the polygon?

36. Let $ABCD$ be a rectangle with $AB = a$ and $BC = b$. Suppose r_1 is the radius of the circle passing through A and B and touching CD; and similarly r_2 in the radius of the circle passing through B and C and touching AD. Show that $r_1 + r_2 \geq (4/8)(a + b)$.

37. Show that $19^{93} - 13^{99}$ is a positive integer divisible by 162.

38. If a, b, c, d are four positive real numbers such that $abcd = 1$, prove that $(1 + a)(1 + b)(1 + c)(1 + d) \geq 16$.

39. In a group of ten persons, each person is asked to write the sum of the ages of all the other nine persons. If all the ten sums form the nine-element set {82, 83, 84, 85, 87, 89, 90, 91, 92}, find

the individual ages of the persons, assuming them to be whole numbers (of years).

40. I have six friends and during a certain vacation I met them during several dinners. I found that I dined with all the six exactly on one day; with every five of them on 2 days; with every four of them on 3 days; with every three of them on 4 days; with every two of them on 5 days. Further every friend was present at seven dinners and every friend was absent at 7 dinners. How many dinners did I have alone?

41. From a rectangular piece of paper a triangular corner is cut off resulting in a pentagon. If the sides of the pentagon have lengths 10, 17, 18, 24 and 39 in *some order* find the sides of the rectangle and the sides of the triangle cut off.

42. Find all positive integer solutions x and n of the equation

$$x^2 + 615 = 2^n.$$

43. Let X be a finite set containing n elements. Find the number of all ordered pairs (A, B) of subsets of X such that neither A is contained in B nor B is contained in A.

44. A trapezoid has perpendicular diagonals and altitude 10. Find the area of the trapezoid if one diagonal has length 13.

45. Find the greatest common divisor of all **even** 6-digit numbers obtained by using each of the digits 1, 2, 3, 4, 5, 6 exactly once.

46. If $x_1, x_2, x_3, \ldots, x_n$ are n distinct positive integers, show that there does not exist a positive integer y satisfying

$$x_1^{x_1} + x_2^{x_2} + x_3^{x_3} + \cdots + x_n^{x_n} = y^y.$$

47. Let $M = p^a q^b$ where p, q are prime numbers and a, b are positive integers. Find the number of pairs (m, n) of positive integers, $1 \leq m \leq n \leq M$, such that m divides n and n divides M.

48. Show that for any triangle ABC, the following inequality is true:

$$a^2 + b^2 + c^2 > \sqrt{3}\max\{|a^2 - b^2|, |b^2 - c^2|, |c^2 - a^2|\},$$

where a, b, c are, as usual, the sides of the triangle.

49. In a quadrilateral $ABCD$, it is given that AB is parallel to CD and the diagonals AC and BD are perpendicular to each other. Show that

$$(a)\quad AD \cdot BC \geq AB \cdot CD; \qquad (b)\quad AD + BC \geq AB + CD.$$

50. Let x, y and z be three distinct real positive numbers. Determine with proof whether or not the three real numbers

$$\left|\frac{x}{y} - \frac{y}{x}\right|, \quad \left|\frac{y}{z} - \frac{z}{y}\right|, \quad \left|\frac{z}{x} - \frac{x}{z}\right|$$

can be the lengths of the sides of a triangle.

51. Let n be a positive integer and $p_1, p_2, p_3, \cdots, p_n$ be n prime numbers all larger than 5 such that 6 divides $p_1^2 + p_2^2 + p_3^2 + \cdots + p_n^2$. Prove that 6 divides n.

52. Prove for every natural number n the following inequality:
$\frac{1}{n+1}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)$.

53. Let ABC be a triangle with $AB = AC$ and $\angle BAC = 30°$. Let A' be the reflection of A in the line BC; B' be the reflection of B in the line CA; C' be the reflection of C in the line AB. Show that A', B', C' form the vertices of an equilateral triangle.

54. Prove that the inradius of a right-angled triangle with integer sides is an integer.

55. Show that there are infinitely many pairs (a, b) of relatively prime integers (not necessarily positive) such that both the quadratic equations

$$x^2 + ax + b = 0 \quad \text{and} \quad x^2 + 2ax + b = 0$$

have integer roots.

56. Let ABC be a triangle and a circle Γ' be drawn lying inside the triangle, touching its incircle Γ externally and also touching the two sides AB and AC. Show that the ratio of the radii of the circles Γ' and Γ is equal to $\tan^2(\frac{\pi-A}{4})$.

57. Let C_1 and C_2 be two concentric circles in the plane with radii R and $3R$ respectively. Show that the orthocentre of any triangle inscribed in circle C_1 lies in the *interior* of circle C_2. Conversely, show also that every point in the interior of C_2 is the orthocentre of some triangle inscribed in C_1.

58. If a, b, c are three distinct real numbers and

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a} = t$$

for some real number t prove that $abc + t = 0$.

59. Let a and b be two positive rational numbers such that $\sqrt[3]{a} + \sqrt[3]{b}$ is also a rational number. Prove that $\sqrt[3]{a}$ and $\sqrt[3]{b}$ themselves are rational numbers.

60. Suppose a, b and c are three real numbers such that the quadratic equation

$$x^2 - (a + b + c)x + (ab + bc + ca) = 0$$

has roots of the form $\alpha \pm i\beta$ where $\alpha > 0$ and $\beta \neq 0$ are real numbers [here $i = \sqrt{-1}$]. Show that
(i) the numbers a, b, c are all positive;
(ii) the numbers $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ form the sides of a triangle.

61. Let ABC be an acute-angled triangle in which D, E, F are points on BC, CA, AB respectively such that AD is perpendicular to BC; $AE = EC$; and CF bisects $\angle C$ internally. Suppose CF meets AD and DE in M and N respectively. If $FM = 2, MN = 1, NC = 3$ find the perimeter of the triangle ABC.

62. Let Γ and Γ' be two concentric circles. Let ABC and $A'B'C'$ be any two equilateral triangles inscribed in Γ and Γ' respectively. If P and P' are any two points on Γ and Γ' respectively, show that

$$P'A^2 + P'B^2 + P'C^2 = A'P^2 + B'P^2 + C'P^2.$$

63. Given any four positive, distinct, real numbers, show that one can choose three numbers, say,A, B, C from among them such that all the three quadratic equations

$$Bx^2 + x + C = 0, \quad Cx^2 + x + A = 0, \quad Ax^2 + x + B = 0$$

have only real roots **or** all the three equations have only imaginary roots.

64. The incircle of triangle ABC touches the sides BC, CA and AB in K, L and M respectively. The line through A and parallel to LK meets MK in P and the line through A and parallel to MK meets LK in Q. Show that the line PQ bisects the sides AB and AC of triangle ABC.

65. If a, b, c, x are real numbers such that $abc \neq 0$ and

$$\frac{xb + (1-x)c}{a} = \frac{xc + (1-x)a}{b} = \frac{xa + (1-x)b}{c},$$

then prove that $a = b = c$ or $a + b + c = 0$..

Hints and solutions

3 Use the given equation repeatedly.

4 Use Cauchy-Schwarz inequality.

5 Use the AM-GM inequality for the relations between the coefficients and the zeroes of a polynomial.

6 If $a_n = x + (1/x)$, then $a_{n+1} = a_n \cdot a_1 - a_{n-1}$ for $n > 1$; [Ans:2.]

7 $(-1, -1, -1)$ is the only solution.

8 4 Hrs, $26\frac{122}{143}$ Mins.

9 $(x, y, z) = (2/3, 27/8, 32/3)$.

10 Look for equal numbers in the sequence.

11 When n is odd there are more odd than even numbers in the sequence $1, 2, 3, ..., n$.

12 There is only one such point.

13 49.

15 $(x, y, z) = (2, 3, 1,), (-1, 3, -2)$.

16 There are 8 (obvious) ways.

18 3.

21 $a = 1989, 1993$.

22 $n = -8, -11$.

24 Use the AM-GM inequality.

32 Consider ratios of areas; $[6, \infty)$ and $[8, \infty)$.

33 Drop perpendicular from the midpoint of BC onto AB; ans. 5.

35 640.

36 Express r_1, r_2 in terms of a and b.

37 $19 = 18 + 1$, $13 = 12 + 1$.

39 5, 6, 7, 7, 8, 10, 12, 13, 14, 15.

40 One.

41 The sides of the triangle form a Pythagorean triple. Ans. Rectangle: 39×18; Triangle: 17,8,15.

42 Show that n is even and factorize. Ans.: $(n, x) = (12, 59)$.

43 For every subset A of X, a subset B of X such that $A \not\subset B$ or $B \not\subset A$ can be obtained by taking the union of a proper subset of A with a nonempty subset of $X \setminus A$. Ans.: $4^n - 2 \cdot 3^n + 2^n$.

44 $845/\sqrt{69}$.

45 6.

46 Each $x_j < y$. If $x = \max\{x_j : 1 \leq j \leq n\}$, then $x \geq n$ and hence $y > n$.

47 $(a + 1)(a + 2)(b + 1)(b + 2)/4$.

48 Assume $a \geq b \geq c$ and proceed.

49 Let O be the intersection of two diagonals. Express sides in terms of OA, OB, OC, OD.

50 Assume $x > y > z > 0$ and check the triangle inequality.

51 Any prime > 3 is either of the form $6k - 1$ or of the form $6k + 1$.

52 Use induction on n.

53 If $AB = AC = u$, prove that $(B'C')^2 = (C'A')^2 = (A'B')^2 = 2u^2$.

54 $r = (s - a) \tan \angle A/2$.

55 The given quadratics have integer roots if and only if $a^2 - 4b$ and $4a^2 - 4b$ are both squares. One class of solutions is $a = 2k + 1$, $b = -(k-1)k(k+1)(k+2)$, where 3 does not divide a.

56 The line joining the centres of Γ and Γ' passes through A and bisects $\angle A$.

57 Where does the orthocentre lie?

58 Show that $t = \pm 1$ and determine the value of abc in each case.

59 $a + b = (a^{1/3} + b^{1/3})(a^{2/3} + a^{1/3}b^{1/3} + b^{2/3})$.

60 What is the discriminant of the given quadratic equation? Look for a factorization of this discriminant.

61 Show that D is the mid-point of BC and ABC is equilateral. Ans.: $12\sqrt{3}$.

62 Use Appolonius' theorem in triangle $PB'C'$ and Stewart's theorem in $PA'M$.

63 Assume $0 < x_1 < x_2 < x_3 < x_4$ are distinct real numbers. Then $x_1x_2 < x_1x_3 < x_2x_3 < x_2x_4 < x_3x_4$.

64 $MAEK$ is an isosceles trapezoid.

65 If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \lambda$, then $\lambda = \dfrac{a+c+e}{b+d+f}$.

Appendix B : References

1. S.L. Greitzer, *International Mathematical Olympiads* 1959-77 (compiled with solutions).

2. M.S. Klamkin, *International Mathematical Olympiads* 1978-85 (compiled with solutions).

3. M.S. Klamkin, U.S.A. Mathematical Olympiads 1972-85 (compiled and edited)

   (Indian edition distributed by 'The Mathematical Sciences Trust Soc.,' C-766, New Friends Colony, New Delhi 110 065.)

4. I. Niven and H.S. Zuckerman, *An Introduction to the Theory of Numbers* (third edition), Wiley Eastern Ltd., New Delhi (1991).

5. David M. Burton, *Elementary Number Theory*, Universal Book Stall, New Delhi.

6. S. Barnard and J.M.Child, *Higher Algebra*, Macmillan India Ltd.

7. I.R. Sharygin, *Problems in Plane Geometry*, MIR Publishers, Moscow.

8. D.O. Shklyarsky, N. N. Chentsov and I.M. Yaglom, *Selected Problems and Theorems in elementary Mathematics*, MIR Publishers, Moscow.

9. S.L. Loney, *Plane Trigonometry*, Macmillan India Ltd. (There are many other Indian editions available in the market.)

10. M.R. Modak (Editor), *An excursion in Mathematics*, Bhaskaracharya Pratishthana, 56/14, Erandavane, Damle Path, Pune 411 004.

11. D. Fomin, S. Genkin, and I. Itenberg, *Mathematical Circles*, Universities Press (1998).

12. V. Krishnamurthy, C.R. Pranesachar, K.N. Ranganathan, and B.J. Venkatachala *Challenge and Thrill of Pre-College Mathematics*, New Age Publications (1996).

13. V.K. Balakrishnan, *Combinatorics*, Schaum Series, McGraw-Hill Inc.

14. Howard Eve, *College Geometry*, Narosa Publications.

15. J.N. Kapur, 1000 *Mathematical Challenges*, Mathematical Sciences Trust Society, New Delhi (1996).

Appendix C: Regional Co-ordinators

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